PERIYAR UNIVERSITY

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Salem - 636011, Tamilnadu, India.

CENTRE FOR DISTANCE AND ONLINE EDUCATION (CDOE)

MASTER OF BUSINESS ADMINISTRATION (MBA)

SEMESTER–II

CORE–:APPLIED OPERATIONS RESEARCH

(Candidates admitted from 2024 onwards)

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2024 admission onwards

Core: APPLIED OPERATIONS RESEARCH

Prepared By:

1. **Dr.R.MIYALVAGANAN**
   AssociateProfessor
   DepartmentofManagementStudies
   Vivekandha Institute of Information and ManagementStudies,Elayamplayam,
   Thiruchengode–637205.

2. **Dr.V. JOTHI FRANCINA**
   Assistant Professor
   Department of Management studies
   Sona College of Technology,
   Salem–636005.

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Unit-I

INTRODUCTION TO OPERATIONS RESEARCH

OBJECTIVE: To provide the students with Introduction on OR and its models and to aid in understanding its applicability in the various functional areas of management

1.1 Introduction

Operations Research is a powerful tool that combines mathematical rigor with practical applications to solve complex decision-making problems. By exploiting quantitative techniques and analytical thinking, OR enables organizations to optimize their operations, enhance productivity, and achieve strategic goals. As industries continue to evolve and face new challenges, the role of operations research will remain critical in driving informed decision-making and fostering innovation.

1.2 Overview of Operations Research

Operations Research (OR) is a discipline that applies advanced analytical methods to help make better decisions. It involves the use of mathematical models, statistical analysis, and optimization techniques to solve complex problems in various fields, including business, engineering, healthcare, logistics, and more. The primary goal of operations research is to improve efficiency, productivity, and decision-making processes

OR is the scientific method of providing the executive with an analytical and objective basis for decisions.- P. M. S. Blackett

OR is the art of giving bad answers to problems to which otherwise worse answers are given. - T. L. Saaty

OR is a systematic method oriented study of the basic structures, characteristics, functions and relationships of an organization to provide the executive with a sound, scientific and quantitative basis for decisionmaking.- E. L. Arnoff and M. J. Netzorg

OR is a scientific approach to problem solving for executive management. - H. M. Wagner

OR is an aid for the executive in making his decisions by providing him with the quantitative information based on the scientific method of analysis.- C. Kittee

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1.3 Key Components of Operations Research:

Mathematical Modeling: OR begins with the formulation of mathematical models that represent real-world systems. These models can include linear programming, integer programming, nonlinear programming, and dynamic programming, among others. The choice of model depends on the nature of the problem being addressed.

Optimization: One of the core functions of operations research is to find the best possible solution from a set of feasible solutions. This involves maximizing or minimizing an objective function, such as cost, time, or resource utilization, subject to certain constraints.

Statistical Analysis: OR utilizes statistical methods to analyze data and assess variability and uncertainty in decision-making processes. Techniques such as regression analysis, hypothesis testing, and simulation are commonly employed to draw insights from data.

Simulation: Simulation techniques allow researchers to model complex systems and evaluate the impact of different scenarios without disrupting actual operations. This is particularly useful in systems where analytical solutions are difficult to obtain.

Decision Analysis: OR incorporates decision-making frameworks that help evaluate different alternatives based on their potential outcomes and associated risks. This includes tools like decision trees and multi-criteria decision analysis.

Applications of Operations Research:

Operations research is widely applied across various industries, including:

Supply Chain Management: Optimizing inventory levels, transportation routes, and production schedules to enhance efficiency and reduce costs.

Healthcare: Improving patient flow, resource allocation, and scheduling in hospitals and clinics to enhance service delivery. **Finance:** Portfolio optimization, risk management, and financial forecasting to support investment decisions.

Manufacturing: Streamlining production processes, quality control, and workforce management to maximize output and minimize waste.

Telecommunications: Network design, traffic management, and resource allocation to improve service quality and reduce operational costs.

1.4 Origin of Operations Research (OR)

Operations Research (OR) originated during World War II when military operations required efficient resource allocation and strategic planning. The need for effective decision-making in complex military operations led to the development of mathematical models and analytical techniques to optimize logistics, troop movements, and supply chains.

The term "Operations Research" was first used in the early 1940s by a group of scientists and military personnel in the United Kingdom, who sought to apply scientific methods to military problems. This interdisciplinary approach combined elements of mathematics, engineering, economics, and management science.

After the war, the principles and techniques developed for military applications were adapted for use in various civilian sectors, including manufacturing, transportation, healthcare, and finance. The post-war period saw a significant expansion of OR as organizations recognized the value of data-driven decision-making and optimization in improving efficiency and effectiveness.

1.5 Scope of Operations Research

The scope of Operations Research is broad and encompasses a wide range of applications across various industries. Key areas within the scope of OR include:

Optimization: Finding the best solution from a set of feasible options, such as minimizing costs or maximizing profits. Techniques include linear programming, integer programming, and nonlinear programming.

Statistical Analysis: Analyzing data to understand variability and uncertainty. This includes methods such as regression analysis, hypothesis testing, and quality control.

Simulation: Modeling complex systems to evaluate the impact of different scenarios. Simulation allows for experimentation without disrupting actual operations, making it useful in fields like manufacturing and healthcare.

Decision Analysis: Providing frameworks for evaluating alternatives based on potential outcomes and risks. Tools such as decision trees and multi-criteria decision analysis help in making informed choices.

Supply Chain Management: Optimizing the flow of goods and services from suppliers to customers. This includes inventory management, transportation logistics, and production scheduling.

Healthcare Operations: Improving patient care through better resource allocation, scheduling, and process optimization in hospitals and clinics.

Finance and Investment: Assisting in portfolio optimization, risk assessment, and financial forecasting to support investment decisions.

Manufacturing and Production: Streamlining production processes, managing quality control, and optimizing workforce allocation to enhance productivity.

Telecommunications: Designing networks, managing traffic, and optimizing resource allocation to improve service delivery and reduce costs.

Public Policy and Government: Assisting in policy formulation, resource allocation, and program evaluation to enhance public services and governance.

1.6 Nature of Operations Research:

Interdisciplinary Approach: OR integrates techniques from various fields such as mathematics, statistics, engineering, economics, and computer science to solve complex decision-making problems.

Problem-Solving Focus: The primary goal of OR is to provide solutions to real-world problems, often involving optimization, resource allocation, and logistics.

Quantitative Analysis: OR relies heavily on quantitative data and mathematical models to analyze problems and predict outcomes, ensuring that decisions are based on empirical evidence.

Systematic Methodology: OR employs a systematic approach to problem-solving, which includes defining the problem, formulating a model, analyzing the model, and interpreting the results.

Decision Support: OR provides tools and frameworks that assist decision-makers in evaluating different scenarios and making informed choices.

1.7 Characteristics of Operations Research:

Operations Research (OR) is a discipline that applies advanced analytical methods to help make better decisions. Here are the key characteristics and nature of Operations Research:

Optimization	OR often seeks to find the best possible solution from a set of feasible alternatives, maximizing or minimizing an objective function (e.g., cost, time, or profit).
. Modeling	It involves creating mathematical models that represent complex systems, allowing for simulation and analysis of different variables and constraints
Use of Algorithms	OR utilizes various algorithms and computational techniques to solve optimization problems, including linear programming, integer programming, and simulation methods
Sensitivity Analysis	OR includes the assessment of how changes in input variables affect outcomes, helping to understand the robustness of solutions
Real-World Applications	OR is applied in various industries, including manufacturing, transportation, healthcare, finance, and supply chain management, demonstrating its versatility and practical relevance.
Collaborative Nature	OR often involves collaboration with stakeholders to ensure that the models and solutions developed are relevant and applicable to the specific context of the problem

1.8 Models in OR

Operations Research (OR) employs various models to analyze complex decision-making problems and optimize outcomes. Here are some of the key types of models used in OR:

What is a model? A model is a physical or symbolic representation of the relevant aspects of the reality or system with which one is concerned. In other words, a model is a means of portraying the system or reality of concern to the decision-maker. As such, the concept of a model generally implies a series of connected and identifiable relationships that essentially demonstrate the proposition: if this action, then this result. Thus, 'the model, being an abstraction of the assumed real system, identifies the pertinent relationships of the system in the form of an objective and a set of constraints.'

1. Linear Programming (LP) Models:

Description: These models involve a linear objective function that needs to be maximized or minimized, subject to a set of linear constraints.

Applications : Resource allocation, production scheduling, and transportation problems

2. Integer Programming (IP) Models:

Description: Similar to linear programming, but some or all decision variables are constrained to take on integer values.

Applications Scheduling, facility location, and capital budgeting.

3. Mixed-Integer Programming (MIP) Models:

Description: A combination of linear programming and integer programming, where some variables are continuous and others are integers

Applications: Complex scheduling and logistics problems

- :.**4. Dynamic Programming (DP) Models:**

Description: These models break down a problem into simpler subproblems and solve them recursively, often used for problems that involve decision-making over time

Applications: Inventory management, resource allocation over time, and shortest path problems

..**5. Network Models:**

Description:These models represent problems in terms of networks (nodes and arcs) and are used to analyze flow through a network.

Applications Transportation, telecommunications, and project management (e.g., PERT/CPM)

6. Simulation Models:

Description: These models use random sampling and statistical methods to simulate the behavior of complex systems over time.

Applications: Queuing systems, inventory systems, and risk analysis.

- - **7. Queuing Models:**

Description: These models analyze waiting lines or queues to optimize service efficiency and minimize wait times.

Applications: Customer service, telecommunications, and manufacturing processes.

8. Game Theory Models:

Description: These models study strategic interactions among rational decision-makers, focusing on competitive situations

Applications: Economics, military strategy, and negotiation scenarios.

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- **9. Markov Decision Processes (MDP):**

Description: These models are used for decision-making in situations where outcomes are partly random and partly under the control of a decision-maker.

Applications: Inventory control, robotics, and finance.

- **10. Heuristic and Metaheuristic Models:**

Description: These are approximate methods used for solving complex optimization problems where traditional methods may be inefficient or infeasible.

Applications: Traveling salesman problem, scheduling, and vehicle routing.

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1.9 Applications of Operations Research in Functional areas of Management

Operations Research (OR) has a wide range of applications across various functional areas of management. Here are some key applications in different domains:

1. Production and Operations Management:

- **Production Scheduling**: OR techniques help optimize production schedules to minimize downtime and maximize output.
- **Inventory Management**: Models such as Economic Order Quantity (EOQ) and Just-In-Time (JIT) help manage inventory levels efficiently, reducing holding costs and stockouts.
- **Quality Control**: Statistical process control and optimization models are used to maintain and improve product quality.

2. Supply Chain Management:

- **Logistics Optimization**: OR models optimize transportation routes and distribution networks to minimize costs and improve delivery times.
- **Supplier Selection**: Decision-making models assist in evaluating and selecting suppliers based on cost, quality, and reliability.
- **Demand Forecasting**: OR techniques help in predicting customer demand, enabling better planning and resource allocation.

2. Marketing Management:

OR techniques equally help the marketing management to determine (i) where distribution points and warehousing should be located, their size, quantity to be stocked and the choice of customers; (ii) the optimum allocation of sales budget to direct selling and promotional expenses; (iii) the choice of different media of advertising and bidding strategies; and (iv) the consumer preferences relating to size, colour, packaging, and so forth, for various products as well as to outbid and outwit competitors.

- **Market Research**: OR methods analyze consumer data to identify trends and preferences, aiding in targeted marketing strategies.
- **Pricing Strategies**: Optimization models help determine optimal pricing strategies to maximize revenue and market share.
- **Advertising Campaigns**: Simulation and modeling techniques evaluate the effectiveness of different advertising channels and campaigns.

4. Financial Management:

In the field of finance, the progress of operations research has been rather slow. Financial institutions have many situations similar to those prevailing in other commercial organizations. Investment policy must maximize the return on it, keeping the factor of risk below a specified level. Long-range corporate objectives of an institution are always studied along with functional objectives of individual departments, consistency between the two being essential for corporate planning. In particular, banking institutions require precise and accurate forecasting on cash management and capital budgeting. Linear programming models for many problems and quadratic programming formulation for some complex problems may prove the usefulness of operations research techniques to credit institutions

- **Portfolio Optimization**: OR techniques are used to construct investment portfolios that maximize returns while minimizing risk.
- **Capital Budgeting**: Decision analysis models assist in evaluating investment projects and determining the best allocation of financial resources.

- **Risk Management**: OR methods help in assessing and mitigating financial risks through various modeling techniques.

5. Human Resource Management:

- **Workforce Scheduling**: OR models optimize employee schedules to meet operational needs while considering labor laws and employee preferences.
- **Recruitment and Selection**: Decision-making models assist in evaluating candidates based on various criteria to select the best fit for the organization.
- **Training and Development**: OR techniques help assess training needs and evaluate the effectiveness of training programs.

6. Project Management:

- **Project Scheduling**: Techniques like PERT (Program Evaluation and Review Technique) and CPM (Critical Path Method) are used to plan and control project timelines.
- **Resource Allocation**: OR models optimize the allocation of resources across multiple projects to ensure timely completion and cost-effectiveness.
- **Risk Analysis**: Simulation and sensitivity analysis help identify potential risks in project execution and develop mitigation strategies.

7. Healthcare Management:

The application of operations research techniques can as well be noticed in the context of hospital management, health planning programmes, transportation system and in several other sectors. For instance, hospital management quite often faces the problem of allotting its limited resources, of its multiphased activities. Optimum allocation of resources, so as to ensure a certain desirable level of service to patients, can be reached through OR. Similarly, for operating a transportation service profitably, an optimum schedule for vehicles and crew members becomes necessary. Punctuality, waiting time, total travel time, speed of the vehicles and agreement with trade unions of crew members are some of the variables to be taken into account while preparing optimum schedules, which can be worked out utilizing OR techniques.

As stated earlier, operations research is generally concerned with problems that are tactical rather than strategic in nature, that is, its use to long-range organizational planning problems is very much limited. However, one can hope that with further developments taking place in the field, operations research will be able to deal with organizations in their entirety, rather than with slices through them. This description brings home the point that during the last four decades, the scope of OR has extensively been widened and hopefully shall attain new strides by the end of the present century. The art of systems analysis, so well developed in the military context, will spread to other contexts—notably such civil government branches as criminal justice, urban problems, housing, health, care, education and social services. There is growing awareness amongst the people that unless they make themselves familiar with OR techniques, they would not be able to understand and appreciate the problems of modern business units. With computer facilities becoming widespread the significance and scope of OR is likely to grow in the coming years. In fact, the future holds great promise for the growth of OR in power, scope and practical importance and utility. OR will continue its currently vigorous efforts to reach out to new arenas of exploration and application.

- **Patient Flow Optimization**: OR techniques improve patient scheduling and flow in hospitals to reduce wait times and enhance service delivery.
- **Resource Allocation**: Models assist in the optimal allocation of medical staff, equipment, and facilities to improve healthcare outcomes.

- **Healthcare Operations**: Simulation models evaluate the impact of different operational strategies on patient care and resource utilization.

Let us Sum up:

Operations Research has evolved into a vital discipline that leverages quantitative methods to solve complex problems across various sectors. Its origins in military applications have paved the way for its widespread adoption in civilian industries, where it continues to play a crucial role in enhancing decision-making, optimizing operations, and driving efficiency. As organizations face increasingly complex challenges, the scope of OR will continue to expand, incorporating new methodologies and technologies to address emerging needs. Operations Research is a powerful tool for decision-making that combines analytical rigor with practical application, making it essential for organizations seeking to optimize their operations and improve efficiency. Each of the models serves a specific purpose and is chosen based on the nature of the problem being addressed. By applying these models, organizations can make informed decisions that enhance efficiency and effectiveness in their operations. Operations Research provides valuable tools and methodologies that enhance decision-making and operational efficiency across various functional areas of management. By leveraging OR techniques, organizations can optimize processes, reduce costs, and improve overall performance.

Check your Progress

1. Identify the key features of operations research.
2. In what ways does operations research serve as a tool for scientific analysis?
3. How is operations research applied in the field of hospital management?

SELF ASSESSMENT QUESTIONS

Short-Answer Questions

1. Write a short note on how operations research techniques are used in modern business and industrial units.
2. What is the role of operations research techniques in business and industry?
3. Write a brief note on the various models in operations research

Long-Answer Questions

1. Define operations research. Explain some of its features.
2. Discuss the scope of operations research
3. Is it possible to make all business decisions with the assistance of operations research? Give detailed reasons to support your answer.
4. Enumerate the advantages of modelling in OR.
5. ‘Modelling enables quick and economical experimentation for finding an optimum solution to a given problem.’ Do you agree? Give reasons for your answer.

Multiple Choice Questions

1. Operations research is the application of_____methods to arrive at the optimal Solutions to the problems.
   - A. economical
   - **B. scientific**
   - C. a and b both
   - D. artistic
2. In operations research, the are prepared for situations.
   - **A. mathematical models**
   - B. physical models diagrammatic
   - C. diagrammatic models
3. Operations management can be defined as the application of ------------------ -to a problem within a system to yield the optimal solution.
   - A. Suitable manpower
   - **B. mathematical techniques, models, and tools**

C. Financial operations

4. Operations research is based upon collected information, knowledge and advanced study of various factors impacting a particular operation. This leads to more informed -----------

   A. Management processes

   B. Decision making

   C. Procedures

5. OR can evaluate only the effects of........

   A. Personnel factors.

   B. Financial factors

   C. Numeric and quantifiable factors.

True-False

6. By constructing models, the problems in libraries increase and cannot be solved.

   A. True

   B. False

7. Operations Research started just before World War II in Britain with the establishment of teams of scientists to study the strategic and tactical problems involved in military operations.

   A. True

   B. False

8. OR can be applied only to those aspects of libraries where mathematical models can be prepared.

   A. True

   B. False

9. The main limitation of operations research is that it often ignores the human element in the production process.

   A. True

   B. False

10. Which of the following is not the phase of OR methodology?

A. Formulating a problem

B. Constructing a model

C. Establishing controls

D. Controlling the environment

11. The objective function and constraints are functions of two types of variables, _____variables and variables.

A. Positive and negative

B. Controllable and uncontrollable

C. Strong and weak

D. None of the above

12. Operations research was known as an ability to win a war without really going in to ___

A. Battle field

B. Fighting

C. The opponent

D. Both A and B

13. Who defined OR as scientific method of providing executive departments with a quantitative basis for decisions regarding the operations under their control?

A. Morse and Kimball (1946)

B. P.M.S. Blackett (1948)

C. E.L. Arnoff and M.J. Netzorg

D. None of the above

14. OR has a characteristics that it is done by a team of

A. Scientists

B. Mathematicians

C. Academics

D. All of the above

15. Hungarian Method is used to solve

A. A transportation problem

B. A travelling salesman problem

C. A LP problem

D. Both a & b

16. A solution can be extracted from a model either by

A. Conducting experiments on it

B. Mathematical analysis

C. Both A and B

D. Diversified Techniques

17. OR uses models to help the management to determine its _

A. Policies

B. Actions

C. Both A and B

D. None of the above

18. What have been constructed from OR problems an methods for solving the models that are available in many cases?

A. Scientific Models

B. Algorithms

C. Mathematical Models

D. None of the above

19. Which technique is used in finding a solution for optimizing a given objective, such as profit maximization or cost reduction under certain constraints?

A. Quailing Theory

B. Waiting Line

C. Both A and B

D. Linear Programming

20. What enables us to determine the earliest and latest times for each of the events and activities and thereby helps in the identification of the critical path?

A. Programme Evaluation

B. Review Technique (PERT)

C. Both A and B

D. Deployment of resources

21. OR techniques help the directing authority in optimum allocation of various limited resources like____

A. Men and Machine

B. Money

C. Material and Time

D. All of the above

22. The Operations research technique which helps in minimizing total waiting and service costs is

A. Queuing Theory

B. Decision Theory

C. Both A and B

D. None of the above

GLOSSARY

OR: Operations Research.

MS: Management Science.

Symbolic Model: An abstract model, generally using mathematical symbols.

Criterion: is measurement, which is used to evaluation of the results.

Integer Programming: is a technique, which ensures only integral values of variables in the problem.

Dynamic Programming: is a technique, which is used to analyze multistage decision process.

Linear Programming: is a technique, which optimizes linear objective function under limited constraints.

Inventory Model: these are the models used to minimize total inventory costs.

Optimization: Means maximization or minimization

Unit II

LINEAR PROGRAMMING PROBLEM

OBJECTIVE : To understand the concept of Linear Programming models in determining profit maximisation and cost minimisation

2.1 INTRODUCTION TO LINEAR PROGRAMMING

Linear Programming is a decision-making methodology designed to assist managers in their choices. This mathematical approach is utilized to identify the most efficient distribution of resources in order to achieve specific goals, particularly when resources such as finances, labor, materials, machinery, and other assets can be utilized in various ways.

Decision-making is a critical aspect of the business and industrial sectors, particularly when it comes to the challenges associated with the production of goods. Production managers frequently face key questions such as: (i) What products should be manufactured? (ii) In what quantities and through which methods should they be produced? Alfred Marshall, a prominent British economist, emphasized that entrepreneurs analyze their production functions and input costs, often substituting one input for another until they achieve the lowest possible costs. According to Marshall, this substitution largely relies on the businessman’s instinctive judgment rather than on formal calculations.

However, there exists a systematic approach known as Linear Programming, which was initially developed by Russian mathematician L.V. Kantorovich and later refined in 1947 by American mathematician George B. Dantzig for the complex scheduling needs of the U.S. Air Force. Today, Linear Programming is applied to a broad array of real-world business challenges. The rise of electronic computing has further broadened its application, enabling solutions to a variety of industrial issues. It is now regarded as one of the most adaptable management techniques available.

2.2 THE NATURE OF LINEAR PROGRAMMING PROBLEM

Two of the most common are:

1. The product-mix problem
2. The blending Problem

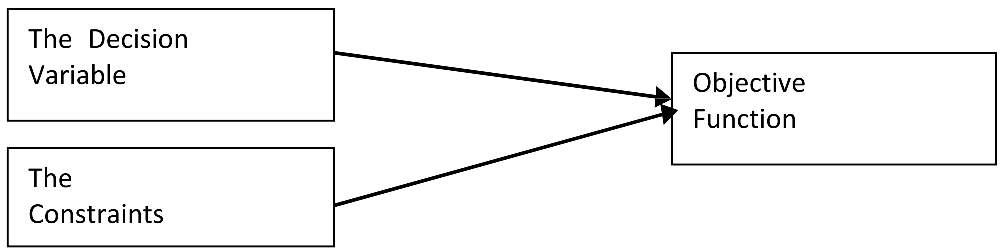
The product mix dilemma involves multiple products, often referred to as candidates or activities, vying for a finite set of resources. The challenge lies in identifying which products to incorporate into the production plan and in what amounts they should be manufactured or sold to optimize profits, enhance market share, or increase sales revenue.

On the other hand, the blending issue pertains to the selection of the most effective combination of available ingredients to create a specific quantity of a product while adhering to stringent specifications. The optimal blend is defined as the least expensive combination of necessary inputs.

2.3 FORMULATION OF THE LINEAR PROGRAMMING MODEL

Three components are:

1. The decision variable
2. The environment (uncontrollable) parameters
3. The result (dependent) variable



Linear Programming Model is composed of the same components

2.4 TERMINOLOGY USED IN LINEAR PROGRAMMING PROBLEM

1. **Components of LP Problem:** Every LPP is composed of a. Decision Variable, b. Objective Function, c. Constraints.

2. Optimization: Linear Programming attempts to either maximise or minimize the values ofthe objective function.

3. Profit of Cost Coefficient: The coefficient of the variable in the objective function express the rate at which the value of the objective function increases or decreases by including in the solution one unit of each of the decision variable.

4. Constraints: The maximisation (or minimisation) is performed subject to a set of constraints. Therefore LP can be defined as a constrained optimisation problem. They reflect the limitations of the resources.

5. Input-Output coefficients: The coefficient of constraint variables are called the Input- Output Coefficients. They indicate the rate at which a given resource is unitized or depleted. They appear on the left side of the constraints.

6. Capacities: The capacities or availability of the various resources are given on the right hand side of the constraints.

2.5 THE MATHEMATICAL EXPRESSION OF THE LP MODEL

The general LP Model can be expressed in mathematical terms as shown below:Let O_{ij} = Input-Output Coefficient

C_j = Cost (Profit) Coefficient

b_i = Capacities (Right Hand Side)

X_j = Decision Variables

Find a vector (x_1, x_2, x_3 ....x_n) that minimise or maximise a linear objective function F(x) where $F(x) = c_1x_1 + c_2x_2 + \quad + c_nx_n$

subject to linear constraints

$a_1x_1 + a_2x_2 + \quad + a_nx_n = b_2$

$a_1x_1 + a_2x_2 + \quad + a_nx_n \le b_2$

....

....

....

$am_1x_1 + am_2x_2 +$ ....+ $am_nx_n \le b_2$

and non-negativity constraints

....................$x_1 \ge 0$, $x_2 \ge 0$, , $x_n \ge 0$

FORMULATION OF LPP

STEPS

1. Identify decision variables
2. Write objective function
3. Formulate constraints

EXAMPLE 1. (PRODUCTION ALLOCATION PROBLEM)

A firm produces three products. These products are processed on three different machines. The time required to manufacture one unit of each of the three products and the daily capacityof the three machines are given in the table below:

Machine	Time per unit (Minutes)			Machine Capacity (minutes/day)
	Product 1	Product 2	Product 3	
M_1	2	3	2	440
M_2	4	-	3	470
M_3	2	5	-	430

It is required to determine the daily number of units to be manufactured for each product. Theprofit per unit for product 1, 2 and 3 is Rs. 4, Rs.3 and Rs.6 respectively. It is assumed that all the amounts produced are consumed in the market. Formulate the mathematical (L.P.) model that will maximise the daily profit.

Formulation of Linear Programming

ModelStep 1

From the study of the situation find the key-decision to be made. In the given situation keydecision is to decide the extent of products 1, 2 and 3, as the extents are permitted to vary.

Step 2

Assume symbols for variable quantities noticed in step 1. Let the extents (amounts) ofproducts 1, 2 and 3 manufactured daily be x_1, x_2 and x_3 units respectively.

Step 3

Express the feasible alternatives mathematically in terms of variable. Feasible alternatives are those which are physically, economically and financially possible. In the given situation feasible alternatives are sets of values of x_1, x_2 and x_3 units respectively. where x_1, x_2 and $x_3 \geq 0$. since negative production has no meaning and is not feasible.

Step 4

Mention the objective function quantitatively and express it as a linear function of variables. In the present situation, objective is to maximize the profit.

i.e., $Z = 4x_1 + 3x_2 + 6x_3$

Step 5

Put into words the influencing factors or constraints. These occur generally because of constraints on availability (resources) or requirements (demands). Express these constraints also as linear equations/inequalities in terms of variables. Here, constraints are on the machine capacities and can be mathematically expressed as

$$2x_1 + 3x_2 + 2x_3 \leq 440$$

$$4x_1 + 0x_2 + 3x_3 \leq 470$$

$$2x_1 + 5x_2 + 0x_3 \leq 430$$

EXAMPLE 2: PRODUCT MIX PROBLEM

A factory manufactures two products A and B. To manufacture one unit of A, 1.5 machine hours and 2.5 labour hours are required. To manufacture product B, 2.5 machine hours and 1.5 labour hours are required. In a month, 300 machine hours and 240 labour hours are available. Profit per unit for A is Rs. 50 and for B is Rs. 40. Formulate as LPP.

Solution:

Products	**Resource/unit**	
	Machine	**Labour**
A	1.5	2.5
B	2.5	1.5
Availability	300 hrs	240 hrs

There will be two constraints. One for machine hours availability and for labour hoursavailability.

Decision variables

X_1 = Number of units of A manufactured per month.

X_2 = Number of units of B manufactured per month

The objective function:

$$\text{Max } Z = 50x_1 + 40x_2$$

Subjective Constraints

For machine hours

$$1.5x_1 + 2.5x_2 \leq 300$$

For labour hours

$$2.5x_1 + 1.5x_2 \leq 240$$

Non negativity

$$x_1, x_2 \geq 0$$

EXAMPLE 3: MEDIA SELECTION

An advertising agency is planning to launch an ad campaign. Media under consideration are T.V., Radio & Newspaper. Each medium has different reach potential and different cost. Minimum 10, 000, 000 households are to be reached through T.V. Expenditure on newspapers should not be more than Rs. 10, 00, 000. Total advertising budget is Rs. 20 million.

Following data is available:

Medium	Cost per Unit(Rs.)	Reach per unit (No. of households)
Television	2, 00, 000	20, 00, 000
Radio	80, 000	10, 00, 000
Newspaper	40, 000	2, 00, 000

Solution:

Decision Variables:

x_1 = Number of units of T.V. ads,

x_2 = Number of units of Radio ads,

x_3 = Number of units of Newspaper ads.

Objective function: (Maximise reach)

Max. Z = 20, 00, 000 x_1 + 10, 00, 000 x_2 + 2, 00, 000x_3

Subject to constraints:

20, 00, 000 $x_1 \geq$ 10, 000, 000 (for T.V.)

40, 000 $x_3 \leq$ 10, 00, 000 (for Newspaper)

2, 00, 000x_1 + 80, 000x_2 + 40, 000$x_3 \leq$ 20, 000, 000 ..(Ad. budget)

$x_1, x_2, x_3 \geq 0$

∴ Simplifying constraints:

$20 x_1 + 8 x_2 + 4 x_3 \leq 2000$

$5 x_1 + 2x_2 + x_3 \leq 500$

$x_1, x_2, x_3 \geq 0$

EXAMPLE 4: DIET PROBLEM

Vitamins B_1 and B_2 are found in two foods F_1 and F_2. 1 unit of F_1 contains 3 units of B_1 nd 4units of B_2. 1 unit of F_2 contains 5 units of B_1 and 3 units of B_2 respectively. Minimum daily prescribed consumption of B_1 & B_2 is 50 and 60 units respectively. Cost perunit of F_1 & F_2 is Rs. 6 & Rs. 3 respectively.

Formulate as LPP.

Solution:

Vitamins	Foods		Minimum Consumption
	F_1	F_2	
B_1	3	5	30
B_2	5	7	40

Decision Variables:

x_1 = No. of units of P_1 per day.

x_2 = No. of units of P_2 per day.

Objective function:

Min. $Z = 100\ x_1 + 150\ x_2$

Subject to constraints:

$3x_1 + 5x_2 \geq 30$ (for N_1)

$5x_1 + 7x_2 \geq 40$ (for N_2)

$x_1, x_2 \geq 0$

Case Study:

1. A manufacturer produces two types of models, M1 and M2 . Each model of the type M1 requires 4 hours of grinding and 2 hours of polishing, whereas each model of the type M2 requires 2 hours of grinding and 5 hours of polishing. The manufacturers have 2 grinders and 3 polishers. Each grinder works 40 hours a week and each polisher works for 60 hours a week. The profit on M1 model is 3.00 and on model M2 is 4.00. Whatever is produced in a week is sold in the market. How should the manufacturer allocate his production capacity to the two types of models, so that he may make the maximum profit in a week?
2. A firm produces two different types of products, Product M and Product N. The firm uses the same machinery for manufacturing both the products. One unit of Product M requires 10 minutes while one unit of Product N requires 2 minutes. The maximum hours the machine can function optimally for a week is 35 hours. The raw material requirement for Product M is 1 kg, while that of Product N is 0.5 kg. Also, the market constraint on product M is 600 kg, while that of Product N is 800 units per week. The cost of manufacturing Product M is 5 per unit and it is sold at 10, while the cost of Product N is 6 per unit and sold at 8 per unit. Calculate the total number of units of Product M and Product N that should be produced per week, so as to derive maximum profit.

2.6 MERITS OF LPP

1. Helps management to make efficient use of resources.
2. Provides quality in decision making.
3. Excellent tools for adjusting to meet changing demands.
4. Fast determination of the solution if a computer is used.
5. Provides a natural sensitivity analysis.
6. Finds solution to problems with a very large or infinite number of possible solution.

2.7 DEMERITS OF LPP

1. Existence of non-linear equation: The primary requirements of Linear Programming is the objective function and constraint function should be linear. Practically linear relationship do not exist in all cases.

2. Interaction between variables: LP fails in a situation where non-linearity in the equation emerge due to joint interaction between some of the activities like total effectiveness.

3. Fractional Value: In LPP fractional values are permitted for the decision variable.

4. Knowledge of Coefficients of the equation: It may not be possible to state all coefficientsin the objective function and constraints with certainty.

Check your Progress

1. Explain what is meant by decision variables, objective function and constraints in LinearProgramming.
2. Give the mathematical formulation of the linear programming problems.
3. What are the components of LPP? What is the significance of non-negativity restriction?
4. State the limitations of LPP.
5. Give the assumptions and advantages of LPP.
6. An investor wants to identify how much to invest in two funds, one equity and one debt. Total amount available is Rs. 5, 00, 000. Not more than Rs. 3, 00, 000 should be invested in a single fund. Returns expected are 30% in equity and 8% in debt. Minimum return on total investment should be 15%. Formulate as LPP.

7. A company manufactures two products P_1 and P_2. Profit per unit for P_1 is Rs. 200 and for P_2 is Rs. 300. Three raw materials M_1, M_2 and M_3 are required. One unit of P_1 needs 5 units of M_1 and 10 units of M_2. One unit of P_2 needs 18 units of M_2 and 10 units of M_3. Availability is 50 units of M_1, 90 units of M_2 and 50 units of M_3. Formulate as LPP.

8. A firm produces two products X and Y. Minimum 50 units of X should be produced. Thereis no limit for producing Y. Profit per unit is Rs. 100 for X and Rs. 150 for Y.

Product	Resource Requirement	Resource Availability
X	20 Machine Hours	Machine Hours = 2500
	10 Labour Hours	Labour Hours = 3000
Y	10 Machine Hours	
	15 Labour Hours	

Formulate as LPP.

9. A patient has been recommended two nutrients N_1 and N_2 everyday. Minimum intake is 10g for N_1 and 15g for N_2 everyday. These nutrients are available in two products P_1 and P_2. One unit of P_1 contains 2g of N_1 and 3g of N_2. One unit of P_2 contains 1g of N_1 and 2g of N_2. Cost per unit is Rs. 200 for P_1 and Rs. 150 for P_2. Formulate as LPP such that nutrient requirement can be fulfilled at the lowest cost.

10. Two vitamins A and B are to be given as health supplements on daily basis to students. There are two products Alpha & Beta which contain vitamins A and B. One unit of Alpha contains 2g of A and 1g of B. One unit of Beta contains 1g of A and 2g of B. Daily requirements for A and B are atleast 10g each. Cost per unit of Alpha is Rs. 20 and of Beta is Rs. 30. Formulate as LPP to satisfy the requirements at minimum cost.

2.8 LINEAR PROGRAMMING SOLUTION - GRAPHICAL METHOD

There are two methods available to find optimal solution to a Linear Programming Problem. One is graphical method and the other is simplex method.Graphical method can be used only for a two variables problem i.e. a problem which involves two decision variables. The two axes of the graph (X & Y axis) represent the two decision variables X_1 & X_2.

GRAPHICAL METHOD OF SOLVING LPP

Step 1: Formulation of LPP (Linear Programming Problem)

Use the given data to formulate the LPP.

Maximisation

Example 1

A company manufactures two products A and B. Both products are processed on two machines M_1 & M_2.

	M_1	M_2
A B	6 Hrs/Unit 4 Hrs/Unit	2 Hrs/Unit 4 Hrs/Unit
Availability	7200 Hrs/month	4000 Hrs/month

Profit per unit for A is Rs. 100 and for B is Rs. 80. Find out the monthly production of A andB to maximise profit by graphical method.

Formulation of LPP

X_1 = No. of units of A/Month

X_2 = No. of units of B/Month

Max Z = 100 X_1 + 80 X_2

Subject to constraints:

$6X_1 + 4X_2 \le 7200$

$2X_1 + 4X_2 \le 4000$

$X_1, X_2 \ge 0$

Step 2: Determination of each axis

Horizontal (X) axis: Product A (X_1)

Vertical (Y) axis: Product B (X_2)

Step 3: Finding co-ordinates of constraint lines to represent constraint lines on thegraph.

The constraints are presently in the form of inequality (≤). We should convert them intoequality to obtain co-ordinates.

Constraint No. 1: $6 X_1 + 4 X_2 \leq 7200$

Converting into equality:

$6 X_1 + 4 X_2 \leq 7200$

X_1 is the intercept on X axis and X_2 is the intercept on Y axis.To find X_1, let $X_2 = 0$

$6 X_1 = 7200$

$6 X_1 = 7200$ ∴ $X_1 = 1200$; $X_2 = 0$ (1200, 0)

To find X_2, let $X_1 = 0$

$4 X_2 = 7200$ $X_2 = 1800$; $X_1 = 0$ (0, 1800)

Hence the two points which make the constraint line are:

(1200, 0) and (0, 1800)

Note: When we write co-ordinates of any point, we always write (X_1, X_2). The value of X_1 iswritten first and then value of X_2. Hence, if for a point X_1 is 1200 and X_2 is zero, then its co-ordinates will be (1200, 0).

Similarly, for second point, X_1 is 0 and X_2 is 1800. Hence, its co-ordinates are (0, 1800).

Constraint No. 2:

$2 X_1 + 4 X_2 \leq 4000$

To find X_1, let $X_2 = 0$

$2 X_1 = 4000$ ∴ $X_1 = 2000$; $X_2 = 0$ (2000, 0)

To find X_2, let $X_1 = 0$

$4 X_2 = 4000$ ∴ $X_2 = 1000$; $X_1 = 0$ (0, 1000)

Each constraint will be represented by a single straight line on the graph. There are twoconstraints, hence there will be two straight lines.

The co-ordinates of points are:

1. Constraint No. 1: (1200, 0) and (0, 1800)
2. Constraint No. 2: (2000, 0) and (0, 1000)

Step 4: Representing constraint lines on graph

To mark the points on the graph, we need to select appropriate scale. Which scale to take willdepend on maximum value of X_1 & X_2 from co-ordinates.

For X_1, we have 2 values → 1200 and 2000

∴Max. value for $X_2 = 2000$

For X_2, we have 2 values → 1800 and 1000

∴Max. value for $X_2 = 1800$

Assuming that we have a graph paper 20 X 30 cm. We need to accommodate our lines suchthat for X-axis, maximum value of 2000 contains in 20 cm.

∴ Scale 1 cm = 200 units

∴ 2000 units = 10 cm (X-axis)

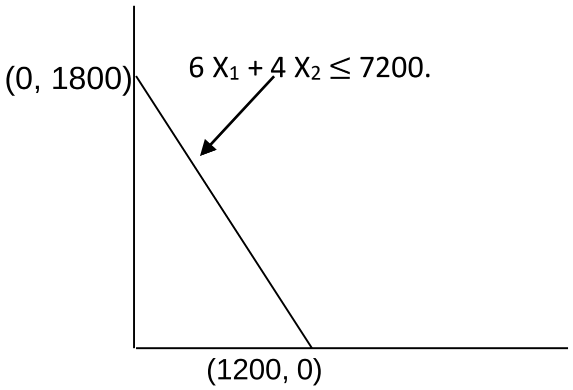
1800 units = 9 cm (Y-axis)

The scale should be such that the diagram should not be too small.

Constraint No. 1:

The line joining the two points (1200, 0) and (0, 1800) represents the constraint$6 X_1 + 4 X_2 \leq 7200$.

Fig 1.



Every point on the line will satisfy the equation (equality) $6 X_1 + 4 X_2 \leq 7200$.

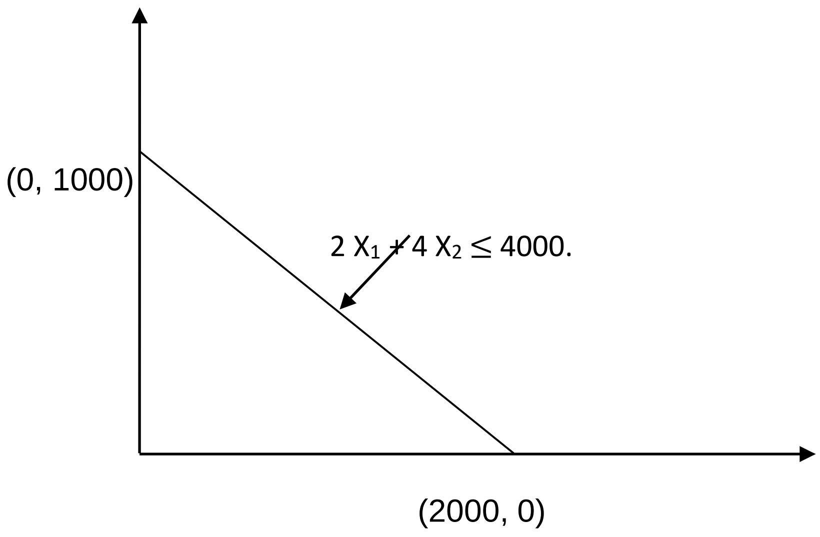
Constraint No. 2:

The line joining the two points (2000, 0) and (0, 1000) represents the constraint $2X_1 + 4X_2 \leq 4000$

Every point on the line will satisfy the equation (equality) $2X_1 + 4X_2 \leq 4000$.

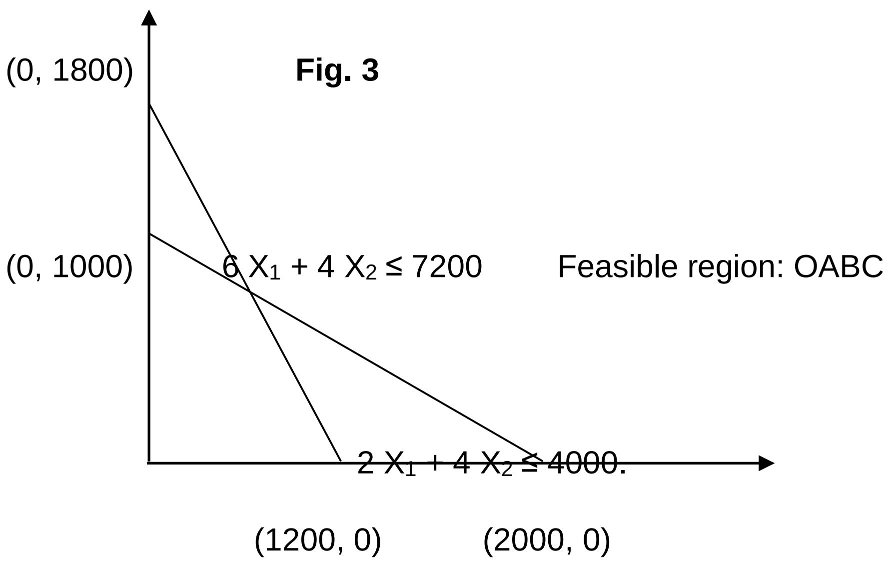
Every point below the line will satisfy the inequality (less than) $2X_1 + 4X_2 \leq 4000$.

Fig 2



Now the final graph will look like this:

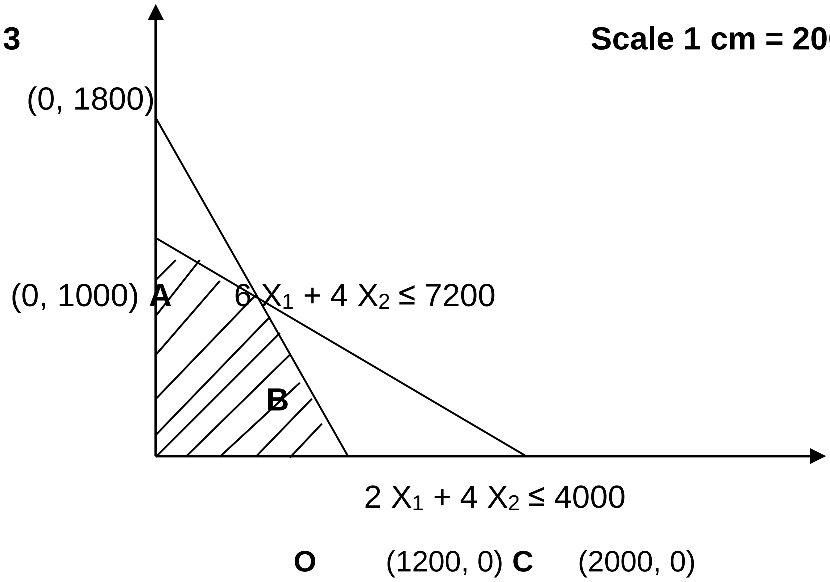
Fig. 3



Step 5: Identification of Feasible Region

The feasible region is the region bounded by constraint lines. All points inside the feasible region or on the boundary of the feasible region or at the corner of the feasible region satisfy all constraints. Both the constraints are 'less than or equal to' (≤) type. Hence, the feasible region should be inside both constraint lines. Hence, the feasible region is the polygon OABC. 'O' is the origin whose coordinates are (0, 0). O, A, B and C are called vertices of the feasible region.

Fig 3 **Scale 1 cm = 200 units**



Step 6: Finding the optimal Solution

The optimal solution always lies at one of the vertices or corners of the feasible region.To find optimal solution:

We use corner point method. We find coordinates ($X_{1,}$ X_2 Values) for each vertex or cornerpoint. From this we fine 'Z' value for each corner point.

Vertex	**Co-ordinates**	**Z = 100 X_1 + 80 X_2**
O	$X_1 = 0$, $X_2 = 0$ From Graph	Z = 0
A	$X_1 = 0$, $X_2 = 1000$ From Graph	Z = Rs. 80, 000
B	$X_1 = 800$, $X_2 = 600$ From Simultaneous equations	Z = Rs. 1, 28, 000
C	$X_1 = 1200$, $X_2 = 0$ From Graph	Z = Rs. 1, 20, 000

Max. Z = Rs. 1, 28, 000 (At point B)

For B ⟶ B is at the intersection of two constraint lines 6 X_1 + 4 $X_2 \leq 7200$ and 2

$X_1 + 4X_2 \le 4000$. Hence, values of X_1 and X_2 at B must satisfy both the equations.

We have two equations and two unknowns, X_1 and X_2.

Solving simultaneously. $6 X_1 + 4 X_2 \le 7200$ (1)

$2 X_1 + 4 X_2 \le 4000$ (2)

$4 X_1 = 3200$ Subtracting (2) from (1)

$X_1 = 800$

Substituting value of X_1 in equation (1), we

get $4 X_2 = 2400$ $\therefore X_2 = 600$

Solution

Optimal Profit = Max Z = Rs. 1, 28, 000 Product Mix:

X_1 = No. of units of A / Month = 800

X_2 = No. of units of A / Month = 600

2.9 MINIMISATION CASE

Example 2

A firm is engaged in animal breeding. The animals are to be given nutrition supplements everyday. There are two products A and B which contain the three required nutrients.

Nutrients	**Quantity/unit**		**Minimum Requirement**
	A	**B**	
1	72	12	216
2	6	24	72
3	40	20	200

Product cost per unit are: A: rs. 40; B: Rs. 80. Find out quantity of product A & B to be givento provide minimum nutritional requirement.

Step 1: Formulation as LPP

X_1 - Number of units of A

X_2 - Number of units of B

Z - Total Cost

Min. $Z = 40 X_1 + 80 X_2$

Subject to constraints:

$72 X_1 + 12 X_2 \ge 216$

$6 X_1 + 24 X_2 \ge 72$

$40\ X_1 + 20\ X_2 \geq 200$

$X_1, X_2 \geq 0.$

Step 2: Determination of each axis

Horizontal (X) axis: Product A (X_1)

Vertical (Y) axis: Product B (X_2)

Step 3: Finding co-ordinates of constraint lines to represent the graph

All constraints are 'greater than or equal to' type. We should convert them into equality:

1. Constraint No. 1: $72\ X_1 + 12\ X_2 \geq 216$

Converting into equality

$72\ X_1 + 12\ X_2 = 216$

To find X_1, let $X_2 = 0$

$72\ X_1 = 216$ $\therefore X_1 = 3, X_2 = 0$ (3, 0)

To find X_2, let $X_1 = 012$

$X_2 = 216$ $\therefore X_1 = 0, X_2 = 18$ (0, 18)

2. Constraint No. 2:

$6\ X_1 + 24\ X_2 \geq 72$

To find X_1, let $X_2 = 0$

$6\ X_1 = 72$ $\therefore X_1 = 12, X_2 = 0$ (12, 0)

To find X_2, let $X_1 = 0$

$24\ X_2 = 72$ $\therefore X_1 = 0, X_2 = 3$ (0, 3)

3. Constraint No. 3:

$40\ X_1 + 20\ X_2 \geq 200$

To find X_1, let $X_2 = 0$

$40\ X_1 = 200$ $\therefore X_1 = 5, X_2 = 0$ (5, 0)

To find X_2, let $X_1 = 020$

$X_2 = 200$ $\therefore X_1 = 0, X_2 = 10$ (0, 10)

The co-ordinates of points are:

1. Constraint No. 1: (3, 0) & (0, 18)

2. Constraint No. 2: (12, 0) & (0, 3)

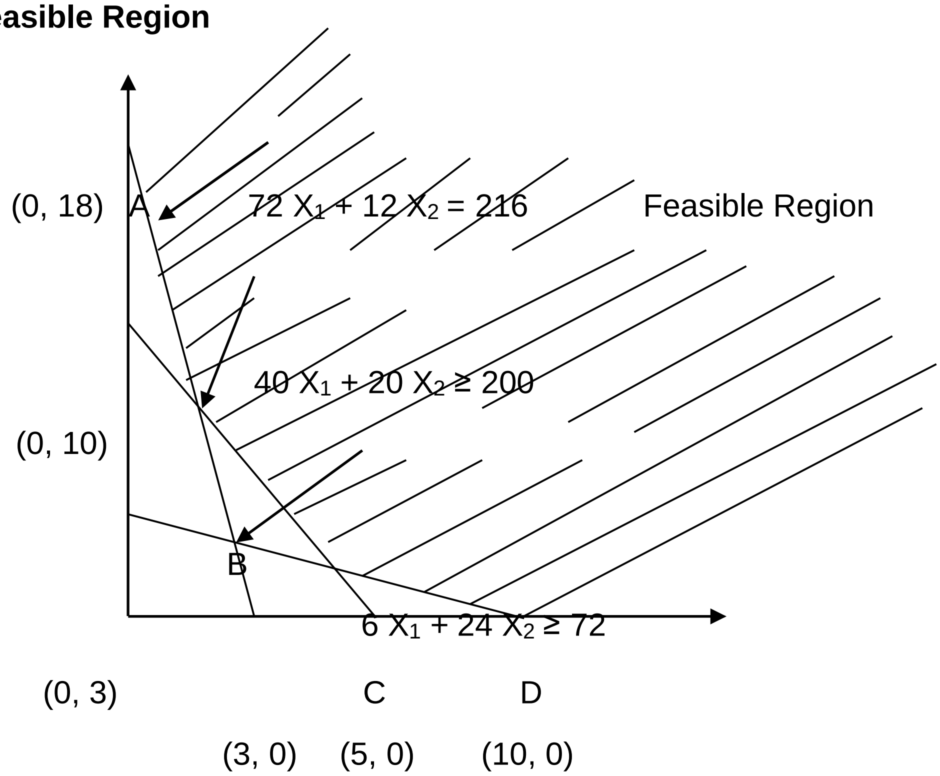
3. Constraint No. 3: (5, 0) & (0, 5)

Every point on the line will satisfy the equation (equality) $72\ X_1 + 12\ X_2 = 216$.

Every point above the line will satisfy the inequality (greater than) $72\ X_1 + 12\ X_2 = 216$.

Similarly, we can draw lines for other two constraints.

Step 5: Feasible Region



All constraints are greater than or equal to (≥) type. Hence, feasible region should be above(to the right of) all constraints. The vertices of the feasible region are A, B, C & D.

Step 6: Finding the optimal solution

Corner Point Method

Vertex	Co-ordinates	$Z = 40\ X_1 + 80\ X_2$
A	$X_1 = 0, X_2 = 18$ From Graph	$\therefore Z = 1, 440$
B	$X_1 = 2, X_2 = 6$ From Simultaneous Equations	$\therefore Z = 560$
C	$X_1 = 4, X_2 = 2$ From Simultaneous Equations	$\therefore Z = 320$
D	$X_1 = 12, X_2 = 0$	$\therefore Z = 480$

	From graph	

∴ Min. Z = Rs. 320 (At point 'C')

For B - Point B is at intersection of constraint lines '$72 X_1 + 12 X_2 \geq 216$' and '$40 X_1 + 20 X_2 \geq 200$'.

Hence, point B should satisfy both the equations.

$72 X_1 + 12 X_2 = 216$ (1)

$40 X_1 + 20 X_2 = 200$ (2)

$\therefore 360 X_1 + 60 X_2 = 1080$ (1) x 5

$120 X_1 + 60 X_2 = 600$ (2) x 3

$\therefore 240 X_1 = 480$

$X_1 = 2$

Substituting value of X_1 in equation (1), we get:

$12 X_2 = 216 - 144 = 72$

$X_2 = 6$

For C - Point C is at intersection of constraint lines '$6 X_1 + 24 X_2 = 72$' and '$40 X_1 + 20 X_2 = 200$'. Hence, point C should satisfy both the equations.

$6 X_1 + 24 X_2 = 72$ (1)

$40 X_1 + 20 X_2 = 200$ (2)

$30 X_1 + 120 X_2 = 360$ (1) x 5

$240 X_1 + 120 X_2 = 1200$ (2) x 6

$210 X_1 = 840$

$X_1 = 4$

Substituting value of X_1 in equation (1),

we get $24 X_2 = 72 - 24 = 48$

$X_2 = 2$

Solution

Optimal Cost = Z min = Rs. 320

Optimal Product Mix:

X_1 = No. of units of product A = 4

X_2 = No. of units of product B = 2

2.10 MAXIMISATION-MIXED CONSTRAINTS

Example 1

A firm makes two products P_1 & P_2 and has production capacity of 18 tonnes per day. P_1 & P_2 require same production capacity. The firm must supply at least 4 t of P_1 & 6 t of P_2 per day. Each tonne of P_1 & P_2 requires 60 hours of machine work each. Maximum machine hours available are 720. Profit per tonne for P_1 is Rs. 160 & P_2 is Rs. 240. Find optimal solution by graphical method.

LPP Formulation

X_1 = Tonnes of P_1 / Day

X_2 = Tonnes of P_2 / Day

Max. $Z = 160\ X_1 + 240\ X_2$

Subject to

constraints $X_1 \geq 4$

$X_2 \geq 6$

$X_1 + X_2 \leq 18$

$60\ X_1 + 60\ X_2 \leq 720$

$X_1, X_2 \geq 0$

Coordinates for constraint lines:

1. $X_1 \geq 4$ — (4, 0).... No value for X_2, $\therefore X_2 = 0$
2. $X_2 \geq 6$ — (0, 6).... No value for X_1, $\therefore X_1 = 0$
3. $X_1 + X_2 \leq 18$ — (18, 0) (0, 18)
4. $60\ X_1 + 60\ X_2 \leq 720$ — (12, 0) (0, 12)

If $X_1 = 0$, $60\ X_2 = 720$ — $\therefore X_2 = 12$ (0, 12)

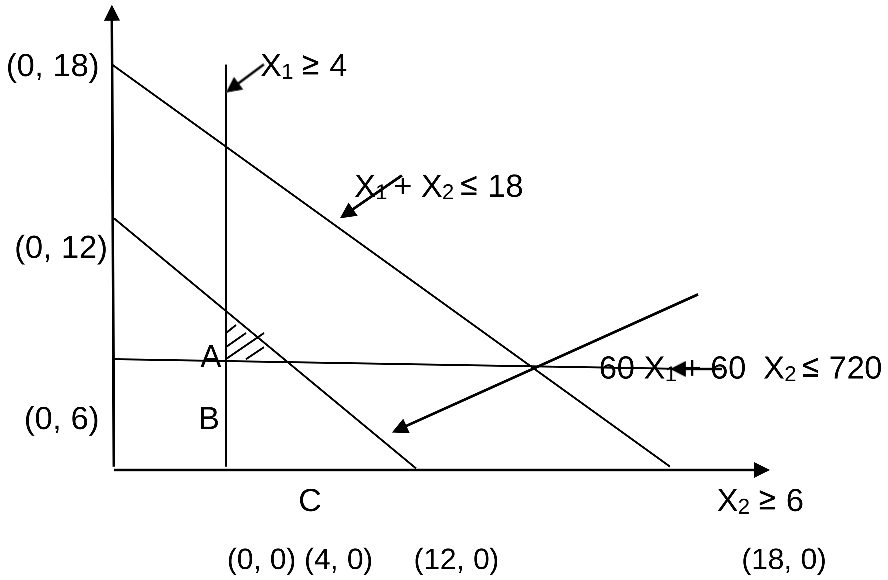
If $X_2 = 0$, $60\ X_1 = 720$ — $\therefore X_1 = 12$ (12, 0)

Graph: X_1: X Axis

X_2: Y Axis

Scale:

Maximum value for X_1 = 18; Maximum value for X_2 = 18; $\therefore$ Scale: 1 cm = 2 Tonnes.



Two constraints are 'greater than or equal to' type. Hence, feasible region will be above or to the rig ht of these constraint lines. Two constraints are 'less than or equal to' type. Hence, feasible region will be below or to the left of these constraint lines. Hence, feasible region is ABC.

Optimal Solution

Corner Point Method

Vertex	Coordinates	$Z = 160 X_1 + 240 X_2$
A	$X_1 = 4, X_2 = 8$ Simultaneous Equation	∴ Z = Rs. 2, 560
B	$X_1 = 4, X_2 = 6$ From Graph	∴ Z = Rs. 2, 080
C	$X_1 = 6, X_2 = 6$ Simultaneous Equations	∴ Z = Rs. 2, 400

For A ⟶ $X_1 = 4$ from graph

A is on the line $60 X_1 + 60 X_2 = 720$

$60 X_2 = 720 - 60 (4) = 480$ $\therefore X_2 = 8$

For C ⟶ $X_2 = 6$ from graph

A is on the line $60 X_1 + 60 X_2 = 720$

$60 X_1 = 720 - 60 (6) = 360$ $\therefore X_1 = 6$

$\therefore$ Z = Rs. 2, 560 (At point 'A')

Solution

Optimal Profit Z. Max = Rs. 2, 560

X_1 = Tonnes of P_1 = 4 tonnes

X_2 = Tonnes of P_2 = 8 tonnes.

2.11 MINIMISATION MIXED CONSTRAINTS

Example 2:

A firm produces two products P and Q. Daily production upper limit is 600 units for total production. But at least 300 total units must be produced every day. Machine hours consumption per unit is 6 for P and 2 for Q. At least 1200 machine hours must be used daily. Manufacturing costs per unit are Rs. 50 for P and Rs. 20 for Q. Find optimal solution for the LPP graphically.

LPP formulation

X_1 = No. of Units of P / Day

X_2 = No. of Units of Q / Day

Min. $Z = 50 X_1 + 20 X_2$

Subject to constraints

$X_1 + X_2 \leq 600$

$X_1 + X_2 \geq 300$

$6 X_1 + 2 X_2 \geq 1200$

$X_1, X_2 \geq 0$

Coordinates for Constraint lines

1. $X_1 + X_2 = 600$

If $X_1 = 0$, $X_2 = 600$ ∴ (0, 600)

If $X_2 = 0$, 60 $X_1 = 600$ ∴ (600, 0)

2. $X_1 + X_2 = 300$

If $X_1 = 0$, $X_2 = 300$ ∴ (0, 300)

If $X_2 = 0$, 60 $X_1 = 300$ ∴ (300, 0)

3. $6 X_1 + 2 X_2 \geq 1200$

If $X_1 = 0$, $2X_2 = 1200$ ∴ $X_2 = 600$ (0, 600)

If $X_2 = 0$, 6 $X_1 = 1200$ ∴ $X_1 = 200$ (200, 0)

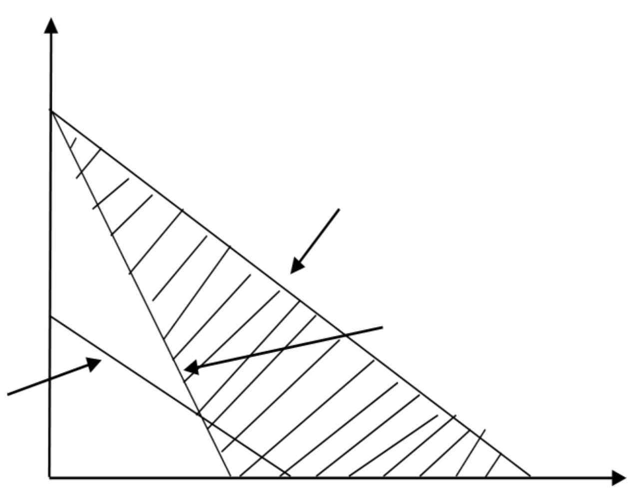
Graph: X_1: X Axis

X_2: Y Axis

Scale:

Maximum value for $X_1 = 600$; Maximum value for $X_2 = 600$; ∴ Scale: 1 cm = 50 units.

Feasible region is ABCD.



$X_1 + X_2 = 600$ $X_1 + X_2 = 300$ $6 X_1 + 2 X_2 \geq 1200$

Two constraints are 'greater than or equal to' type. Hence, feasible region will be above or to the right of these constraint lines. Two constraints are 'less than or equal to' type.

Hence, feasible region will be below or to the left of these constraint lines. Hence, feasible region is ABCD.

Optimal Solution Corner

Point Method

Vertex	Coordinates	$Z = 160 X_1 + 240 X_2$
A	$X_1 = 0, X_2 = 600$ From Graph	$\therefore$ Z = Rs. 12, 000
B	$X_1 = 150, X_2 = 150$ Simultaneous Equations	$\therefore$ Z = Rs. 10, 500
C	$X_1 = 300, X_2 = 0$ From Graph	$\therefore$ Z = Rs. 15, 000
D	$X_1 = 600, X_2 = 0$ From Graph	$\therefore$ Z = Rs. 30, 000

Min. Z = Rs. 10, 500

For B - B is at intersection of two constraint lines '$6 X_1 + 2 X_2 \geq 1200$' and '

$X_1 + X_2 = 300$'

.$6 X_1 + 2 X_2 \geq 1200$ (1)

$X_1 + X_2 = 300$ (2)

$2X_1 + 2X_2 = 600$ (2) X 2

$2X_1 = 600$

$X_1 = 150$

Substituting value in Equation (2), $X_2 = 150$. Solution

Optimal Cost = Rs. 10, 500/

X_1 = No. of Units of P = 150

X_2 = No. of Units of P = 150.

Check your progress

1. What is meant by feasible region in graphical method.
2. What is meant by 'iso-profit' and 'iso-cost line' in graphical solution.
3. Mr. A. P. Ravi wants to invest Rs. 1, 00, 000 in two companies 'A' and 'B' so as not to exceed Rs. 75, 000 in either of the company. The company 'A' assures average return of 10% in whereas the average return for company 'B' is 20%. The

risk factor rating of company 'A' is 4 on 0 to 10 scale whereas the risk factor rating for 'B' is 9 on similar scale. As Mr. Ravi wants to maximise his returns, he will not accept an average rate of return below 12% risk ora risk factor above 6.
Formulate this as LPP and solve it graphically.

4. Solve the following LPP graphically and interpret the result.

Max. $Z = 8X_1 + 16 X_2$
Subject to:
$X_1 + X_2 \leq 200$
$X_2 \leq 125$
$3X_1 + 6X_2 \leq 900$
$X_1, X_2 \geq 0$

5. A furniture manufacturer makes two products - tables and chairs. Processing of these products is done on two types of machines A and B. A chair requires 2 hours on machine type A and 6 hours on machine type B. A table requires 5 hours on machine type A and no time on Machine type B. There are 16 hours/day available on machine type A and 30 hours/day on machine type B. Profits gained by the manufacturer from a chair and a table are Rs. 2 and Rs. 10 respectively. What should be the daily production of each of the two products? Use graphical method of LPP to find the solution.

2.12 Special Cases in Linear Programming

a. Infeasible Solution (Infeasibility)

Infeasible means not possible. Infeasible solution happens when the constraints have contradictory nature. It is not possible to find a solution which can satisfy all constraints.

In graphical method, infeasibility happens when we cannot find Feasible region.

Example 1:

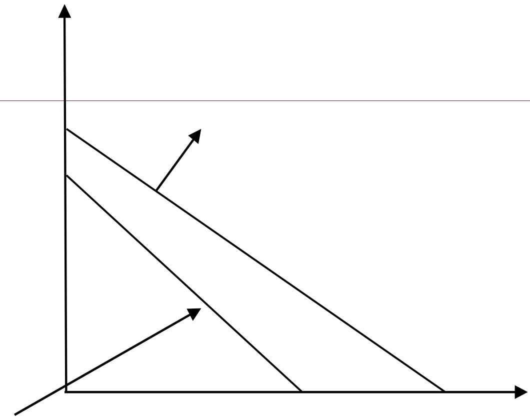
Max. $Z = 5 X_1 + 8 X_2$

Subject to constraints

$4 X_1 + 6 X_2 \leq 24$

$4 X_1 + 8 X_2 \leq 40$ $X_1,$

$X_2 \geq 0$

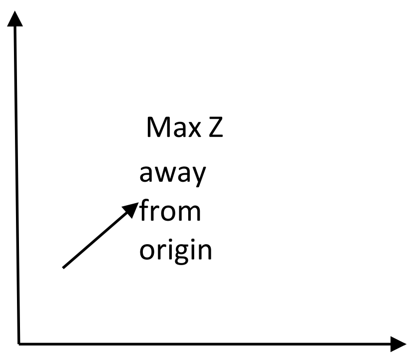


There is no common feasible region for line. Hence, solution is infeasible.

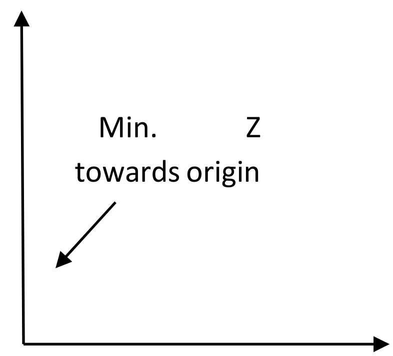
b. Unbounded Solution (Unboundedness)

Unbounded mean infinite solution. A solution which has infinity answer is called unbounded solution.

In graphical solution, the direction with respect to origin is as follows:

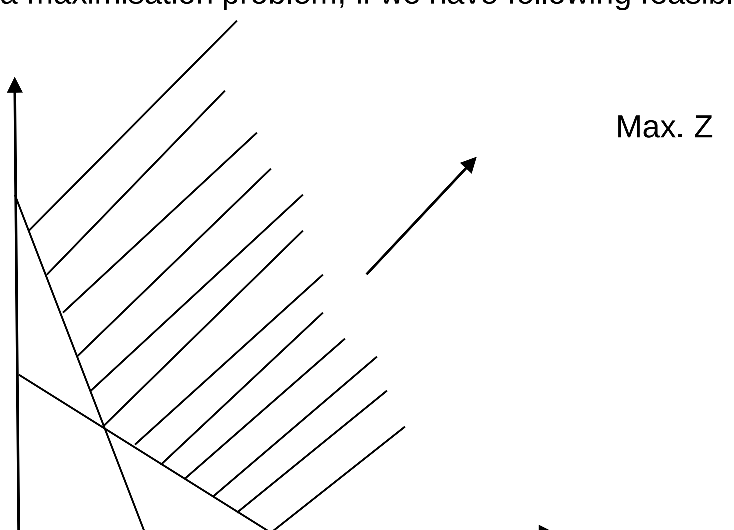


Maximisation



Minimisation

Now, in a maximisation problem, if we have following feasible region:



There is no upper limit (away from origin), hence the answer will be infinity. This is called unbounded solution.

c. Redundant Constraint (Redundancy)

A constraint is called redundant when it does not affect the solution. The feasible region doesnot depend on that constraint.

Even if we remove the constraint from the solution, the optimal answer is not affected.

Example

Max. $Z = 5 X_1 + 8 X_2$

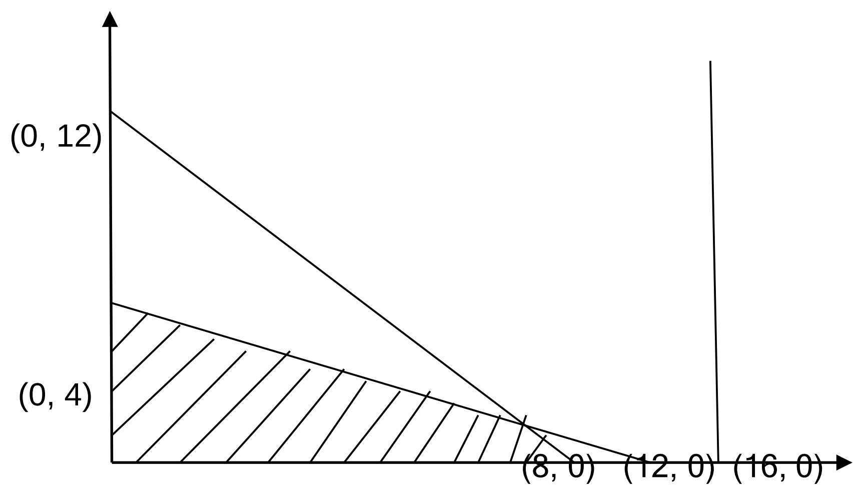
Subject to Constraints

$3 X_1 + 2 X_2 \leq 24$

$X_1 + 3 X_2 \leq 12$

$X_1 \leq 16$

$X_1, X_2 \geq 0$



The feasible region for the above problem is OABC. The 3rd constraint does not affect thefeasible region. Hence, the constraint $X_1 \leq 16$ is redundant constraint.

d. Alternate Optimal Solution: (Multiple Optimal Solution)

Alternate or multiple optimal solution means a problem has more than one solution whichgives the optimal answer.

There are two or more sets of solution values which give maximum profit or minimum cost. In graphical method, we come to know that there is optimal solution which is alternative

2.13 Concept of Shadow Price

Shadow price of resource means value of one extra unit of resource. It is the maximum price the company should pay for procuring extra resources from market. It also indicates profitability or profit contribution of each resource (per unit). Shadow price = 'Z_i' value of slack variables. S_1 - slack variable of resource 1 and S_2 - Slack Variable of resource 2. A slack variable represents unutilised capacity of a resource. Slack Variable is represented by 'S'.

Concept of Duality

Every linear programming problem has a mirror image associated with it. If the original problem is maximisation, the mirror image is minimisation and vice versa. The original problem is called 'primal' and the mirror image is called 'dual'. The format of simplex method is such that when we obtain optimal solution of any one out of primal or dual, we automatically get optimal solution of the other.

For example, if we solve dual by simplex method, we also get optimal solution of primal.

Characteristics of Dual Problem

1. Dual of the dual is primal.
2. If either the primal or dual has a solution, then the other also has a solution. The optimalvalue of both the solutions is equal.
3. If any of the primal or dual is infeasible then the other has an unbounded solution.

Advantages of Duality

1. If primal problem contains a large number of rows and a smaller number of

columns wecan reduce the computational procedure by converting into dual.

2. Solution of the dual helps in checking computational accuracy of the primal.

3. Economic interpretation of the dual helps the management in decision making.

For example,

Minimisation LPP can be solved by two methods:

1. Simplex of Dual Method and

2. Artificial Variable Method Method:1

Simplex of Dual Method

The original problem is called 'Primal'. We convert the problem in its 'Dual'.

Primal	Dual
1. Minimisation Problem Min. Z	Maximisation Problem Max. Z^*
2. Constraints are of ' ≥ ′ type	Constraints are of '≤' type.
3. Decision variables are X_1, X_2 etc.	Decision variables are Y_1, Y_2 etc.

Objective function coefficients of primal (4, 3) become RHS of constraints in Dual.

RHS of Constraints of Primal (4000, 50, 1400) become objective function coefficients of Dual. In the LHS (left side) of constraints, all vertical values are written horizontally in Dual.. No. of Decision Variables in Primal = No. of constraints in Dual

No. of Constraints in Primal = No. of Decision Variables in Dual

For example,

Primal	Dual
Min $Z = 4 X_1 + 3 X_2$ Subject to: $200 X_1 + 100 X_2 \geq 4000$ $1 X_1 + 2 X_2 \geq 50$ $40 X_1 + 40 X_2 \geq 1400$	Max $Z = 4000 Y_1 + 50 Y_2 + 1400 Y_3$ Subject to: $200 y_1 + 1y_2 + 40y_3 \leq 4$ $100 y_1 + 2y_2 + 40y_3 \leq 3$

The numerical is calculated as shown in Simplex Method.

Check your Progress

1. Why an optimal solution to an unbounded maximisation LPP cannot be found in SimplexMethod?
2. What is meant by shadow price of a resource?
3. What do you mean by optimal solution?
4. What is the difference between feasible solution and basic feasible solution?

Multiple Choice Questions

1. What is the objective function in linear programming problems?
   A. A constraint for available resource
   B. An objective for research and development of a company
   C. A linear function in an optimization problem
   D. A set of non-negativity conditions

2. Which statement characterizes standard form of a linear programming problem?
   A. Constraints are given by inequalities of any type
   B. Constraints are given by a set of linear equations
   C. Constraints are given only by inequalities of >= type
   D. Constraints are given only by inequalities of <= type

3. Feasible solution satisfies ________
   A. Only constraints
   B. only non-negative restriction
   C. [a] and [b] both
   D. [a],[b] and Optimum solution

4. In Degenerate solution value of objective function_________.
A. increases infinitely
B. basic variables are nonzero
C. decreases infinitely
D. One or more basic variables are zero

5. Minimize Z = _____________
   A. –maximize(Z)
   B. -maximize(-Z)

C. maximize(-Z)
D. none of the above

6. In graphical method the restriction on number of constraint is_____.
   A. 2
   B. not more than 3
   C. 3
   D. none of the above

7. In graphical representation the bounded region is known as____region.
   A. Solution
   B. basic solution
   C. feasible solution
   D. optimal

8. Graphical optimal value for Z can be obtained from
   A. Corner points of feasible region
   B. Both a and c
   C. corner points of the solution region
   D. none of the above

9. In LPP the condition to be satisfied is
   A. Constraints have to be linear
   B. Objective function has to be linear
   C. none of the above
   D. both a and b

State True or False:

10. Objective function in Linear Programming problems has always finite value at the optimal solution-TRUE
11. A finite optimal solution can be not unique- FALSE
12. Feasible regions are classified into bounded, unbounded, empty and multiple: TRUE
13. Corner points of a feasible region are located at the intersections of the region andcoordinate axes: TRUE

14. Identify the type of the feasible region given by the set of inequalitiesx - y <= 1
x - y >= 2
where both x and y are positive.

A. A triangle
B. A rectangle
C. An unbounded region
D. An empty region

15. Consider the given vectors: a(2,0), b(0,2), c(1,1), and d(0,3). Which of the followingvectors are linearly independent?

A. a, b, and c are independent
B. a, b, and d are independent
C. a and c are independent
D. b and d are independent

16. Consider the linear equation2
x1 + 3 x2 - 4 x3 + 5 x4 = 10
How many basic and non-basic variables are defined by this equation?

A. One variable is basic, three variables are non-basic
B. Two variables are basic, two variables are non-basic
C. Three variables are basic, one variable is non-basic
D. All four variables are basic

17. The objective function for a minimization problem is given byz
= 2 x1 - 5 x2 + 3 x3
The hyperplane for the objective function cuts a bounded feasible region in the space (x1,x2,x3). Find the direction vector d, where a finite optimal solution can be reached.

A. d(2,-5,3)
B. d(-2,5,-3)
C. d(2,5,3)
D. d(-2,-5,-3)

18. The feasible region of a linear programming problem has four extreme points: A(0,0), B(1,1), C(0,1), and D(1,0). Identify an optimal solution for minimization problem with the objective function z = 2 x - 2 y

A. A unique solution at C
B. A unique solutions at D
C. An alternative solution at a line segment between A and B
D. An unbounded solution

19. Degeneracy occurs when

A. Basic variables are positive but some of non-basic variables have negative values
B. The basic matrix is singular, it has no inverse
C. Some of basic variables have zero values
D. Some of non-basic variables have zero values

20. Linear programming is a
(a) Constrained optimization technique
(b) Technique for economic allocation of limited resources.
(c) Mathematical techniques
(d) All of the above

21. Constraints in an LP model represents
(a) Limitation
(b) Requirements
(c) Balancing limitations and requirements
(d) All of the above

22. The distinguished feature of an LP model is
(a) Relationship among all variable is linear
(b) It has single objective function and constraints
(c) Value of decision variables is non-negative
(d) All of the above

23. Alternative solution exist of an LP model when
(a) One of the constraints is redundant
(b) Objective function equation is parallel to one of the
(c) Two constrains are parallel
(d) All of the above

24. In the optimal simplex table, Cj -Zj value indicates
(a) Unbounded solution
(b) Cycling
(c) Alternative solution
(d) None of these

25. For a maximization problem, objective function coefficient for an artificial variable is
(a) + M
(b) -M
(c) Zero
(d) None of these

26. If an optimal solution is degenerate, then
(a) There are alternative optimal solution
(b) The solution is infeasible
(c) The solution is use to the decis ion maker
(d) None of these

27. If a primal LP problem has finite solution, then the dual LP problem should have
(a) Finite solution
(b) Infeasible solution
(c) Unbounded solution
(d) None of these

Let us Sum up

Linear programming (LP) is a strategic approach used for decision-making within specific constraints, based on the premise that the relationships among variables representing different phenomena are linear. LP is one of the most prevalent decision-making methods utilized in business and industry. Its applications span across various domains, including production planning and scheduling, transportation logistics, sales and marketing, financial management, portfolio optimization, and corporate strategy, among others. This technique is particularly effective in addressing resource allocation challenges. Some of its most promising application areas include production planning, transportation, sales and advertising, financial planning, and corporate management. The concept of 'linearity' refers to straightforward, proportional relationships between the relevant variables. In economic theory, linearity corresponds to the principle of constant returns—indicating that when the input quantity is doubled, both the resulting outputs and profits also double. The criterion function, commonly referred to as the objective function, is a vital part of LP problems. It defines whether the goal is to maximize or minimize certain outcome variables. Constraints are the restrictions that shape the planning and decision-making process, outlining the limitations imposed on the decision variables. Feasible solutions encompass all the potential outcomes that can be pursued while adhering to the specified constraints.

SELF ASSESSMENT QUESTIONS

Short Answer Questions

1. What do you mean by linear programming?
2. What is meant by proportionality in linear programming?
3. Write a brief note on certainty in linear programming.
4. What are the basic constituents of an LP model?

Long Answer Questions

1. Analyse the fields where linear programming can be used.
2. Discuss the components of a linear programming problem.
3. Describe the steps used in formulation of a Linear Programming problem (LPP).

2.14 SIMPLEX METHOD

Simplex method is used to find the optimal solution to multivariable problems. The method is basically an algorithm which is used by an examinee to examine corner points in a methodical fashion until he/she arrive at the best solution. In this method, slack variables are added to change the constraints into equations and write all variables to the left of the equal sign and constants to the right.

In real-world scenarios, linear programming issues often involve thousands of variables and are typically solved using computer algorithms. While it is possible to tackle these problems algebraically, this approach is not particularly efficient. For instance, imagine we face a problem with five variables and ten constraints. If we were to examine all possible combinations of equations involving these five unknowns, we would identify all the corner points, assess their feasibility, and ultimately arrive at a solution—if one exists. However, even with such a relatively small set of variables, we might encounter over 250 corner points, making the evaluation of each point a cumbersome task. Therefore, we require a method that employs a systematic algorithm and can be executed on a computer. This method needs to be efficient enough to avoid the necessity of evaluating the objective function for every corner point. Luckily, we have such a method known as the simplex method.

The Simplex Method Algorithm

1. **Set up the problem.** That is, write the objective function and the inequality constraints.
2. **Convert the inequalities into equations.** This is done by adding one slack variable for each inequality.
3. **Construct the initial simplex tableau.** Write the objective function as the bottom row.
4. **The most negative entry in the bottom row identifies the pivot column.**
5. **Calculate the quotients.** The smallest quotient identifies a row. The element in the intersection of the column identified in step 4 and the row identified in this step is identified as the pivot element**.** The quotients are computed by dividing the far right column by the identified column in step 4. A quotient that is a zero, or a negative number, or that has a zero in the denominator, is ignored.
6. **Perform pivoting to make all other entries in this column zero.** This is done the same way as we did with the Gauss-Jordan method.
7. **When there are no more negative entries in the bottom row, we are finished; otherwise, we start again from step 4.**
8. **Read off your answers.** Get the variables using the columns with 1 and 0s. All other variables are zero. The maximum value you are looking for appears in the bottom right hand corner.

Example Niki holds two part-time jobs, Job I and Job II. She never wants to work more than a total of 12 hours a week. She has determined that for every hour she works at Job I, she needs 2 hours of preparation time, and for every hour she works at Job II, she needs one hour of preparation time, and she cannot spend more than 16 hours for preparation. If she makes $40 an hour at Job I, and $30 an hour at Job II, how many hours should she work per week at each job to maximize her income?

Solution

In solving this problem, we will follow the algorithm listed above.

STEP 1. Set up the problem. Write the objective function and the constraints.

Since the simplex method is used for problems that consist of many variables, it is not practical to use the variables x, y, z etc. We use symbols x_1, $x2$, $x3$, and so on.

Let

- x_1= The number of hours per week Niki will work at Job I and
- x_2= The number of hours per week Niki will work at Job II.

It is customary to choose the variable that is to be maximized as Z.

The problem is formulated the same way as we did in the last chapter.

Maximize Subject to: $Z=40x1+30x2$

$x1+x2\leq 12$

$2x1+x2\leq 16$

$x1\geq 0;$

$x2\geq 0$

STEP 2. Convert the inequalities into equations. This is done by adding one slack variable for each inequality.

For example to convert the inequality $x_1+x_2\leq 12$

into an equation, we add a non-negative variable y_1, and we get

$$x_1+x_2+y_1=12$$

Here the variable y_1 picks up the slack, and it represents the amount by which x_1+x_2 falls short of 12.

In this problem, if Niki works fewer than 12 hours, say 10, then y_1 is 2. Later when we read off the final solution from the simplex table, the values of the slack variables will identify the unused amounts.

We rewrite the objective function $Z=40x_1+30x_2$

as $-40x_1-30x_2+Z=0$

After adding the slack variables, our problem reads

Objective function $-40x_1-30x_2+Z=0$

Subject to constraints: $x_1+x_2+y_1=12$

$$2x_1+x_2+y_2=16$$

$x1\geq 0;\ x2\geq 0$

STEP 3. Construct the initial simplex tableau. Each inequality constraint appears in its own row. (The non-negativity constraints do *not* appear as rows in the simplex tableau.) Write the objective function as the bottom row.

Now that the inequalities are converted into equations, we can represent the problem into an augmented matrix called the initial simplex tableau as follows.

x_1	x_2	y_1	y_2	Z	C
1	1	1	0	0	12
2	1	0	1	0	16
–40	–30	0	0	1	0

Here the vertical line separates the left hand side of the equations from the right side. The horizontal line separates the constraints from the objective function. The right side of the equation is represented by the column C.

The reader needs to observe that the last four columns of this matrix look like the final matrix for the solution of a system of equations. If we arbitrarily choose $x_1=0$x1=0 and $x_2=0$x2=0, we get

$$y_1=12 \quad y_2=16 \quad Z=0$$

The solution obtained by arbitrarily assigning values to some variables and then solving for the remaining variables is called the **basic solution** associated with the tableau. So the above solution is the basic solution associated with the initial simplex tableau. We can label the basic solution variable in the right of the last column as shown in the table below.

x_1	x_2	y_1	y_2	Z		
1	1	1	0	0	12	y_1
2	1	0	1	0	16	y_2
–40	–30	0	0	1	0	Z

STEP 4. The most negative entry in the bottom row identifies the pivot column.

The most negative entry in the bottom row is -40; therefore the column 1 is identified.

x_1	x_2	y_1	y_2	Z		
1	1	1	0	0	12	y_1
2	1	0	1	0	16	y_2
–40	–30	0	0	1	0	Z
↑						

Question Why do we choose the most negative entry in the bottom row?

Answer The most negative entry in the bottom row represents the largest coefficient in the objective function - the coefficient whose entry will increase the value of the objective function the quickest.

The simplex method begins at a corner point where all the main variables, the variables that have symbols such as x_1, x_2, x_3 etc., are zero. It then moves from a corner point to the adjacent corner point always increasing the value of the objective function.

In the case of the objective function $Z=40x_1+30x_2$, it will make more sense to increase the value of x_1 rather than x_2. The variable x_1x1 represents the number of hours per week Niki works at Job I. Since Job I pays $40 per hour as opposed to Job II which pays only $30, the variable x_1x1 will increase the objective function by $40 for a unit of increase in the variable x_1x1.

STEP 5. Calculate the quotients. The smallest quotient identifies a row. The element in the intersection of the column identified in step 4 and the row identified in this step is identified as the pivot element.

Following the algorithm, in order to calculate the quotient, we divide the entries in the far right column by the entries in column 1, excluding the entry in the bottom row.

x1	x2	y1	y2	Z			
1	1	1	0	0	12	y1	12 ÷ 1 = 12
[2]	1	0	1	0	16	y2	← 16 ÷ 2 = 8
–40 ↑	–30	0	0	1	0	Z	

The smallest of the two quotients, 12 and 8, is 8. Therefore row 2 is identified. The intersection of column 1 and row 2 is the entry 2, which has been highlighted. This is our pivot element.

STEP 6. Perform pivoting to make all other entries in this column zero.

In chapter 2, we used pivoting to obtain the row echelon form of an augmented matrix. Pivoting is a process of obtaining a 1 in the location of the pivot element, and then making all other entries zeros in that column. our pivot element a 1 by dividing the entire second row by 2. The result follows.

x1	x2	y1	y2	Z	
1	1	1	0	0	12
[1]	1/2	0	1/2	0	8
–40	–30	0	0	1	0

To obtain a zero in the entry first above the pivot element, we multiply the second row by -1 and add it to row 1. We get

x_1	x_2	y_1	y_2	Z	
0	1/2	1	–1/2	0	4
[1]	1/2	0	1/2	0	8
–40	–30	0	0	1	0

To obtain a zero in the element below the pivot, we multiply the second row by 40 and add it to the last row.

x_1	x_2	y_1	y_2	Z		
0	1/2	1	–1/2	0	4	y_1
[1]	1/2	0	1/2	0	8	x_1
0	–10	0	20	1	320	Z

We now determine the basic solution associated with this tableau. By arbitrarily choosing $x_2=0$ and $y_2=0$, we obtain $x_1=8$, $y_1=4$, and $z=320$. If we write the augmented matrix, whose left side is a matrix with columns that have one 1 and all other entries zeros, we get the following matrix stating the same thing.

$x_1=8$, $x_2=0$ $y_1=4$, $y_2=0$ and $z=320$

At this stage of the game, it reads that if Niki works 8 hours at Job I, and no hours at Job II, her profit Z will be $320

STEP 7. When there are no more negative entries in the bottom row, we are finished; otherwise, we start again from step 4.

Since there is still a negative entry, -10, in the bottom row, we need to begin, again, from step 4. This time we will not repeat the details of every step, instead, we will identify the column and row that give us the pivot element, and highlight the pivot element. The result is as follows.

x1	x2	y1	y2	Z			
0	[1/2]	1	–1/2	0	4	y1	← 4 ÷ 1/2 = 8
1	1/2	0	1/2	0	8	x1	8 ÷ 1/2 = 16
0	–10	0	20	1	320	Z	
	↑						

We make the pivot element 1 by multiplying row 1 by 2, and we get

x1	x2	y1	y2	Z	
0	[1]	2	–1	0	8
1	1/2	0	1/2	0	8
0	–10	0	20	1	320

Now to make all other entries as zeros in this column, we first multiply row 1 by -1/2 and add it to row 2, and then multiply row 1 by 10 and add it to the bottom row.

x1	x2	y1	y2	Z		
0	1	2	–1	0	8	x2
1	0	–1	1	0	4	x1
0	0	20	10	1	400	Z

We no longer have negative entries in the bottom row, therefore we are finished.

Question Why are we finished when there are no negative entries in the bottom row?

Answer The answer lies in the bottom row. The bottom row corresponds to the equation:

$$0x_1+0x_2+20y_1+10y_2+Z=400 \text{ or}$$

$$z=400-20y1-10y2$$

Since all variables are non-negative, the highest value Z can ever achieve is 400, and that will happen only when y_1y1 and y_2y2 are zero.

STEP 8. Read off your answers.

We now read off our answers, that is, we determine the basic solution associated with the final simplex tableau. Again, we look at the columns that have a 1 and all other entries zeros. Since the columns labelled y_1 and y_2 are not such columns, we arbitrarily choose $y_1=0$, and $y_2=0$, and we get

The matrix reads $x_1=4$, x2=8 and z=400.

2.14 SPECIAL CASES IN SIMPLEX

1. Unbounded Solution Max Z = $60 X_1 + 20X_2$

Subject to

$2 X_1 + 4X_2 \geq 120$

$8 X_1 + 6X_2 \geq 240 X_1,$

$X_2 \geq 0$

Solution

Max Z = $60 X_1 + 20X_2 + 0S_1 + 0S_2 - MA_1 - MA_2$

Subject to

$2 X_1 + 4X_2 - S_1 + A_1 = 120$

$8 X_1 + 6X_2 - S_2 + A_1 = 240 X_1,$

$X_2, S_1, S_2, A_1, A_2, \geq 0$

When we solve this LPP by simplex method, we will get the following values in 4th SimplexTable.

C_j			60	20	0	0	- M	- M	R.R
C	X	B	X_1	X_2	S_1	S_2	A_1	A_2	
0	S_2	240	0	10	- 4	1			- 60
60	X_1	60	1	2	-1/2	0			- 120
Z_j			60	120	- 30	0			
$\Delta = C_j - Z_j$			0	- 100	30	0			
						↑			

Max Positive $C_j - Z_j = 30$

Key Column = S_1

But there is no positive Replacement Ration R means there is an Entering variable, but thereis no outgoing variable. Hence, the solution is unbounded or infinity. The value of Z (Profit) keeps on increasing infinitely.

2. Infeasible Region Max $Z = 3X_1 + 2X_2$

Subject to:

$X_1 + X_2 \leq 4$

$2X_1 + X_2 \geq 10 X_1,$

$X_2 \geq 0$

Solution Standard Form

Max $Z = 3X_1 + 2X_2 + 0S_1 + 0S_2 - MA_1$

Subject to

$X_1 + X_2 + S_1 = 4$

$2X_1 + X_2 - S_2 + A_1 = 10$

$X_1, X_2, S_1, S_2, A_1 \geq 0$

When we solve this LPP by simplex method, we will get the following values in 2nd SimplexTable.

C_j			60	20	0	0	- M	R.R
C	X	B	X_1	X_2	S_1	S_2	A_1	
3	X_1	4	1	1	1	0	0	
- M	A_1	2	- 2	- 1	-1/2	- 1	1	
Z_j			3	3 + M	3 + 2M	M	- M	

$\Delta = C_j - Z_j$	0	- 1 - M	-3 -2M	- M	0	

No positive Δ value.

All $C_j - Z_j$ values are either zero or negative. Hence, test of optimality is satisfied. So, the solution appears to be optimal. But an artificial variable (A_1) is present in the basis, which has objective function coefficient of - M (infinity). Hence, the solution is infeasible (Not feasible). Infeasibility occurs when there is no solution which satisfies all the constraints of the LPP.

Duality in LPP

Every LPP called the **primal** is associated with another LPP called **dual**. Either of the problems is primal with the other one as dual. The optimal solution of either problem reveals the information about the optimal solution of the other.

Important characteristics of Duality

1. Dual of dual is primal
2. If either the primal or dual problem has a solution then the other also has a solution andtheir optimum values are equal.
3. If any of the two problems has an infeasible solution, then the value of the objectivefunction of the other is unbounded.
4. The value of the objective function for any feasible solution of the primal is less than thevalue of the objective function for any feasible solution of the dual.
5. If either the primal or dual has an unbounded solution, then the solution to the otherproblem is infeasible.
6. If the primal has a feasible solution, but the dual does not have then the primal will nothave a finite optimum solution and vice versa.

Advantages and Applications of Duality

1. Sometimes dual problem solution may be easier than primal solution, particularly whenthe number of decision variables is considerably less than slack / surplus variables.
2. In the areas like economics, it is highly helpful in obtaining future decision

in theactivities being programmed.

3. In physics, it is used in parallel circuit and series circuit theory.
4. In game theory, dual is employed by column player who wishes to minimize his maximum loss while his opponent i.e. Row player applies primal to maximize his minimum gains. However, if one problem is solved, the solution for other also can be obtained from the simplex tableau.
5. When a problem does not yield any solution in primal, it can be verified with dual.
6. Economic interpretations can be made and shadow prices can be determined enabling themanagers to take further decisions.

Steps for a Standard Primal Form

Step 1 – Change the objective function to Maximization form

Step 2 – If the constraints have an inequality sign '≥' then multiply both sides by -1 and convertthe inequality sign to '≤'.

Step 3 – If the constraint has an '=' sign then replace it by two constraints involving theinequalities going in opposite directions. For example $x_1 + 2x_2 = 4$ is written as
$x_1+2x_2 \leq 4$ (using step2) → $-x_1-2x_2 \leq -4$

Step 4 – Every unrestricted variable is replaced by the difference of two non-negative variables.

Step5 – We get the standard primal form of the given LPP in which.

- All constraints have '≤' sign, where the objective function is of maximizationform.
- All constraints have '≥' sign, where the objective function is of minimizationfrom.

Rules for Converting any Primal into its Dual

1. Transpose the rows and columns of the constraint co-efficient.

2. Transpose the co-efficient ($c_1, c_2, \ldots c_n$) of the objective function and the right sideconstants ($b_1, b_2, \ldots b_n$)
3. Change the inequalities from '≤' to '≥' sign.
4. Minimize the objective function instead of maximizing it.

Example

Write the dual of the given problems

Example 1

Min Zx = 2x2 + 5x3

Subject to

x1+x2 ≥ 2

2x1+x2+6x3 ≤ 6

x1 - x2 +3x3 = 4

x1, x2 , x3 ≥ 0

Solution

Primal

Max Zx' = -2x2 – 5x3

Subject to

-x1-x2 ≤ -2

2x1+x2+6x3 ≤ 6

x1 - x2 +3x3 ≤ 4

-x1 + x2 -3x3 ≤ -4

x1, x2 , x3 ≥ 0

Dual

Min Zw = -2w1 + 6w2 + 4w3 – 4w4

Subject to

-w1 + 2w2 +w3 –w4 ≥ 0

-w1 + w2 - w3 +w4 ≥ -2

$6w2 + 3w3 - 3w4 \geq -5$

$w_1, w_2, w_3, w_4 \geq 0$

Example 2

Min $Z_x = 3x_1 - 2x_2 + 4x_3$

Subject to

$3x_1+5x_2 + 4x_3 \geq 7$

$6x_1+x_2+3x_3 \geq 4$

$7x_1 - 2x_2 -x_3 \geq 10$

$x_1 - 2x_2 + 5x_3 \geq 3$

$4x_1 + 7x_2 - 2x_3 \geq 2$

$x_1, x_2 , x_3 \geq 0$

Solution

Primal

Max $Z_x' = -3x_1 + 2x_2 - 4x_3$

Subject to

$-3x_1 - 5x_2 - 4x_3 \leq -7$

$-6x_1 - x_2 - 3x_3 \leq -4$

$-7x_1 + 2x_2 + x_3 \leq - 10$

$-x_1 + 2x_2 - 5x_3 \leq - 3$

$-4x_1 - 7x_2 + 2x_3 \leq - 2$

$x_1, x_2 , x_3 \geq 0$

Dual

Min $Z_w = -7w_1 - 4w_2 - 10w_3 - 3w_4 -2w_5$

Subject to

$-3w_1 - 6w_2 - 7w_3 – w_4 – 4w_5 \geq -3$

$-5w_1 - w_2 + 2w_3 + 2w_4 – 7w_5 \geq 2$

$-4w_1 - 3w_2 + w_3 - 5w_4 + 2w_5 \geq -4$, $w_1, w_2, w_3, w_4, w_5 \geq 0$

Example 3

Max $Z = 2x_1 + 3x_2 + x_3$

Subject to

$4x_1 + 3x_2 + x_3 = 6$

$x_1 + 2x_2 + 5x_3 = 4$

$x_1, x_2 \geq 0$

Solution

Primal

Max $Z_x = 2x_1 + 3x_2 + x_3$

Subject to

$4x_1 + 3x_2 + x_3 \leq 6$

$-4x_1 - 3x_2 - x_3 \leq -6$

$x_1 + 2x_2 + 5x_3 \leq 4$

$-x_1 - 2x_2 - 5x_3 \leq -4$

$x_1, x_2 \geq 0$

Dual

Min $Z_w = 6w_1 - 6w_2 + 4w_3 – 4w_4$

Subject to

$4w_1 - 4w_2 + w_3 – w_4 \geq 2$

$3w_1 - 3w_2 + 2w_3 - 2w_4 \geq 3$

$w_1 - w_2 + 5w_3 - 5w_4 \geq 1$ $w_1, w_2, w_3,$

$w_4 \geq 0$

Example 4

Min $Z_x = x_1 + x_2 + x_3$

Subject to

$x_1 - 3x_2 + 4x_3 = 5$

$x_1 - 2x_2 \leq 32$

$x_2 - x_3 \geq 4$

$x_1, x_2 \geq 0$, x_3 is unrestricted in sign

Solution

Primal

Max $Z' = -x_1 - x_2 - x_3' + x_3''$

Subject to

$x_1 - 3x_2 + 4(x_3' - x_3'') \leq 5$

$-x_1 + 3x_2 - 4(x_3' - x_3'') \leq -5$

$x_1 - 2x_2 \leq 3$

$-2x_2 + x_3' - x_3'' \leq -4$

$x_1, x_2, x_3', x_3'' \geq 0$

Dual

Min $Z_w = 5w_1 - 5w_2 + 3w_3 - 4w_4$

Subject to

$w_1 - w_2 + w_3 \geq -1$

$-3w_1 + 3w_2 - 2w_3 - 2w_4 \geq -1$, $w_2, w_3, w_4, \geq 0$

Example 5

Max $Z = x_1 - x_2 + 3x_3$

Subject to

$x_1 + x_2 + x_3 \leq 10$

$2x_1 – x_3 \leq 2$

$2x_1 - 2x_2 + 3x_3 \leq 6$

$x_1, x_2, x_3 \geq 0$

Solution

Primal

Max $Z_x = x_1 - x_2 + 3x_3$

Subject to

$x_1 + x_2 + x_3 \leq 10$

$2x_1 - x_3 \leq 2$

$2x_1 - 2x_2 + 3x_3 \leq 6$

$x_1, x_2, x_3 \geq 0$

Dual

Min $Z_w = 10w_1 + 2w_2 + 6w_3$

Subject to

$w_1 + 2w_2 +2w_3 \geq 1$

$w_1 - 2w_3 \geq -1$

$w_1 - w_2 + 3w_3 \geq 3$

$w_1, w_2, w_3 \geq 0$

Duality and Simplex Method

1. Solve the given primal problem using simplex method. Hence write the solution of itsdual

Max $Z = 30x_1 + 23x_2 + 29x_3$

Subject to

$6x_1 + 5x_2 + 3x_3 \leq 26$

$4x_1 + 2x_2 + 6x_3 \leq 7x_1 \geq 0,$

$x_2 \geq 0$

Solution

Primal form

Max $Z = 30x_1 + 23x_2 + 29x_3$

Subject to

$6x_1 + 5x_2 + 3x_3 \leq 26$

$4x_1 + 2x_2 + 6x_3 \leq 7x_1 \geq 0,\ x_2 \geq 0$

Max $Z = 30x_1 + 23x_2 + 29x_3 + 0s_1 + 0s_2$

Subject to

$6x_1 + 5x_2 + 3x_3 + s_1 = 26$

$4x_1 + 2x_2 + 6x_3 + s_2 = 7$

$x_1, x_2, s_1, s_2 \geq 0$

		$C_j\rightarrow$	30	23	29	0	0	
Basic Variables	C_B	X_B	X_1	X_2	X_3	S_1	S_2	Min Ratio X_B / X_K
s_1	0	26	6	5	3	1	0	26/6
s_2	0	7	4	2	6	0	1	7/4→
	Z = 0		↑ -30	-23	-29	0	0	$\leftarrow\Delta_j$
s_1	0	31/2	0	2	-6	1	-3/2	31/4
x_1	30	7/4	1	1/2	3/2	0	1/4	7/2→
	Z = 105/2		0	↑ -8	16	0	15/2	$\leftarrow\Delta_j$
s_1	0	17/2	-4	0	-12	1	-5/2	
x_2	23	7/2	2	1	3	0	1/2	
	Z =161/2		16	0	40	0	23/2	$\leftarrow\Delta_j$

$\Delta_j \geq 0$ so the optimal solution is $Z = 161/2$, $x_1 = 0$, $x_2 = 7/2$, $x_3 = 0$.

The optimal solution to the dual of the above problem will be $Z_w^* = 161/2$,

$w_1 = \Delta_4 = 0$, $w_2 = \Delta_5 = 23/2$

In this way we can find the solution to the dual without actually solving it.

2. **Use duality to solve the given problem. Also read the solution of its primal.**

Min $Z = 3x_1 + x_2$

Subject to

$x_1 + x_2 \geq 1$

$2x_1 + 3x_2 \geq 2$

$x_1 \geq 0$, $x_2 \geq 0$

Solution

Primal

Min Z =Max $Z' = -3x_1 - x_2$

Subject to

$- x_1 - x_2 \leq -1$

$-2x_1 - 3x_2 \leq - 2$

$x_1 \geq 0$, $x_2 \geq 0$

Dual

Min $Z_w = -w_1 - 2w_2$

Subject to

$-w_1 - 2w_2 \geq -3$

$-w_1 - 3w_2 \geq -1$

$w_1, w_2 \geq 0$

Changing the dual form to SLPP

Max $Z_w' = w_1 + 2w_2 + 0s_1 + 0s_2$

Subject to

$w_1 + 2w_2 + s_1 = 3$

$w_1 + 3w_2 + s_2 = 1$

$w_1, w_2, s_1, s_2 \geq 0$

	$C_j \rightarrow$	1	2	0	0	
Basic Variables	C_B W_B	W_1	W_2	S_1	S_2	Min Ratio W_B / W_K
s_1 s_2	0 3 0 1	1 1	2 3	1 0	0 1	3/2 1/3←
	$Z_w' = 0$	-1	↑ -2	0	0	←Δ_j
s_1 w_2	0 7/3 2 1/3	1/3 1/3	0 1	1 0	-2/3 1/3	7 1→
	$Z_w' = 2/3$	↑ -1/3	0	0	2/3	←Δ_j
s_1	0 2	0	-1	1	-1	
w_1	1 1	1	3	0	1	
	$Z_w' = 1$	0	1	0	1	←Δ_j

$\Delta_j \geq 0$ so the optimal solution is $Z_w' = 1$, $w_1 = 1$, $w_2 = 0$

The optimal solution to the primal of the above problem will be $Z_x^* = 1$, $x_1 = \Delta_3 = 0$, $x_2 = \Delta_4 = 1$

3. Write down the dual of the problem and solve it.

Max $Z = 4x_1 + 2x_2$Subject to

$- x_1 - x_2 \leq -3$

$-x_1 + x_2 \leq -2$$x_1 \geq 0, x_2 \geq 0$

Solution

Primal

Max $Z = 4x_1 + 2x_2$

Subject to

$- x_1 - x_2 \leq -3$

$-x_1 + x_2 \leq -2$

$x_1 \geq 0, x_2 \geq 0$

Dual

Min $Z_w = -3w_1 - 2w_2$

Subject to

$-w_1 - w_2 \geq 4$

$-w_1 + w_2 \geq 2 w_{1,} w_2 \geq 0$

Changing the dual form to SLPP

Max $Z_w' = 3w_1 + 2w_2 + 0s_1 + 0s_2 - Ma_1 - Ma_2$

Subject to

$-w_1 - w_2 - s_1 + a_1 = 4$

$-w_1 + w_2 - s_2 + a_2 = 2$

$w_1, w_2, s_1, s_2, a_1, a_2 \geq 0$

	$C_j \rightarrow$		3	2	0	0	-M	-M	
Basic Variables	C_B	W_B	W_1	W_2	S_1	S_2	A_1	A_2	Min Ratio W_B / W_K
a_1	-M	4	-1	-1	-1	0	1	0	-
a_2	-M	2	-1	1	0	-1	0	1	2→
	$Z'_w = -6M$		2M - 3	-2 ↑	M	M	0	0	←Δ_j
a_1	-M	6	-2	0	-1	-1	1	X	
w_2	2	2	-1	1	0	-1	0	X	
	$Z'_w = -6M+4$		2M-5	0	M	M-2	0	X	←Δ_j
a_1	-M	6	-2	0	-1	-1	1	X	
w_2	2	2	-1	1	0	-1	0	X	
	$Z'_w =$ -6M+4		2M-5	0	M	M-2	0	X	←Δ_j

$\Delta_j \geq 0$ and at the positive level an artificial vector (a_1) appears in the basis. Therefore the dual problem does not posses any optimal solution. Consequently there exists no finite optimum solution to the given problem.

4. Use duality to solve the given problem.

Min $Z = x_1 - x_2$

Subject to

$2x_1 + x_2 \geq 2$

$-x_1 - x_2 \geq 1$

$x_1 \geq 0$, $x_2 \geq 0$

Solution

Primal

Min Z =Max $Z' = -x_1 + x_2$

Subject to

$- 2x_1 - x_2 \leq -2x_1 + x_2 \leq -1$

$x_1 \geq 0$, $x_2 \geq 0$

Dual

$\text{Min } Z_w = -2w_1 - w_2$

Subject to

$-2w_1 + w_2 \geq -1$

$-w_1 + w_2 \geq 1$

$w_1, w_2 \geq 0$

Changing the dual form to SLPP

$\text{Max } Z'_w = 2w_1 + w_2 + 0s_1 + 0s_2 - Ma$

Subject to

$2w_1 - w_2 + s_1 = 1$

$-w_1 + w_2 - s_2 + a_1 = 1$

$w_1, w_2, s_1, s_2 \geq 0$

Auxiliary LPP

$\text{Max } Z'_w = 0w_1 + 0w_2 + 0s_1 + 0s_2 - 1a_1$

Subject to

$2w_1 - w_2 + s_1 = 1$

$-w_1 + w_2 - s_2 + a_1 = 1$

$w_1, w_2, s_1, s_2, a_1 \geq 0$

Phase I

		$C_j \rightarrow$	0	0	0	0	-1	
Basic Variables	C_B	W_B	W_1	W_2	S_1	S_2	A_1	Min Ratio X_B / X_K
s_1	0	1	2	-1	1	0	0	-
a_1	-1	1	-1	1	0	-1	1	1→
	$Z'_w = -1$		1	-1 ↑	0	1	0	←Δ_j
s_1	0	2	1	0	1	-1	X	
w_2	0	1	-1	1	0	-1	X	

	$Z_w' = 0$	0	0	0	0	X		←Δ_j

$\Delta_j \geq 0$ and no artificial vector appear at the positive level of the basis.

Hence proceed to phase II

Phase II

		C_j→	2	1	0	0	
Basic Variables	C_B	W_B	W_1	W_2	S_1	S_2	Min Ratio X_B / X_K
s_1	0	2	1	0	1	-1	2→
w_2	1	1	-1	1	0	-1	-
	$Z_w' = 1$		-3 ↑	0	0	-1	←Δ_j
w_1	2	2	1	0	1	-1	-
w_2	1	3	0	1	1	-2	-
	$Z_w' = 7$		0	0	3	-4 ↑	←Δ_j

Δ_j = -4 and all the elements of s_2 are negative; hence we cannot find the outgoing vector. This indicates there is an unbounded solution. Consequently by duality theorem the original primal problem will have no feasible solution.

5. Solve the given primal problem using simplex method. Hence write the solution of its dual

Max $Z = 40x_1 + 50x_2$

Subject to

$2x_1 + 3x_2 \leq 3$

$8x_1 + 4x_2 \leq 5x_1 \geq 0, x_2 \geq 0$

Solution

Primal form

Max $Z = 40x_1 + 50x_2$

Subject to

$2x_1 + 3x_2 \leq 3$

$8x_1 + 4x_2 \leq 5$

$x_1 \geq 0, x_2 \geq 0$

SLPP

Subject to

$2x_1 + 3x_2 + s_1 = 3$

$8x_1 + 4x_2 + s_2 = 5$

$x_1, x_2, s_1, s_2 \geq 0$

	$C_j \rightarrow$		40	50	0	0	
Basic Variables	C_B	X_B	X_1	X_2	S_1	S_2	Min Ratio X_B / X_K
s_1	0	3	2	3	1	0	1→
s_2	0	5	8	4	0	1	5/4
	$Z_x = 0$		-40	↑ -50	0	0	←Δ_j
x_2	50	1	2/3	1	1/3	0	3/2
s_2	0	1	16/3	0	-4/3	1	3/16→
	$Z_x = 50$		↑ -20/3	0	50/3	0	←Δ_j
x_2	50	7/8	0	1	1/2	-1/8	
x_1	40	3/16	1	0	-1/4	3/16	
	$Z_x = 205/4$		0	0	15	5/4	←Δ_j

$\Delta_j \geq 0$ so the optimal solution is $Z = 205/4$, $x_1 = 3/16$, $x_2 = 7/8$

The optimal solution to the dual of the above problem will beZ$_w$* = 205/4, $w_1 = \Delta_4 = 15$, $w_2 = \Delta_5 = 5/4$

Additional Worked Examples

Example 1

Minimize $Z = 2x_1 + x_2$

Subject to

$3x_1 + x_2 \geq 3$

$4x_1 + 3x_2 \geq 6$

$x_1 + 2x_2 \geq 3$

and $x_1 \geq 0$, $x_2 \geq 0$

Solution

Step 1 – Rewrite the given problem in the

formMaximize Z2 – = ′$x_1 - x_2$

Subject to

$-3x_1 - x_2 \leq -3$

$-4x_1 - 3x_2 \leq -6$

$-x_1 - 2x_2 \leq -3x_1$,

$x_2 \geq 0$

Step 2 – Adding slack variables to each

constraintMaximize Z2 – = ′$x_1 - x_2$

Subject to

$-3x_1 - x_2 + s_1 = -3$

$-4x_1 - 3x_2 + s_2 = -6$

$-x_1 - 2x_2 + s_3 = -3$

$x_1, x_2, s_1, s_2, s_3 \geq 0$

Step 3 – Construct the simplex table

	$C_j \rightarrow$		-2	-1	0	0	0	
Basic variables	C_B	X_B	X_1	X_2	S_1	S_2	S_3	
s_1	0	-3	-3	-1	1	0	0	
s_2	0	-6	-4	-3	0	1	0	→ outgoing
s_3	0	-3	-1	-2	0	0	1	
	Z0 = ′		2	↑ 1	0	0	0	←Δ_j

Step 4 – To find the leaving vector

Min (-3, -6, -3) = -6. Hence s_2 is outgoing vector

Step 5 – To find the incoming vector

Max (Δ_1 / x_{21}, Δ_2 / x_{22}) = (2/-4, 1/-3) = -1/3. So x_2 is incoming vector

Step 6 –The key element is -3. Proceed to next iteration

	$C_j \rightarrow$		-2	-1	0	0	0	
Basic variables	C_B	X_B	X_1	X_2	S_1	S_2	S_3	
s_1	0	-1	-5/3	0	1	-1/3	0	→ outgoing
x_2	-1	2	4/3	1	0	-1/3	0	
s_3	0	1	5/3	0	0	-2/3	1	
	Z2- = ′		↑ 2/3	0	0	1/3	0	←Δ_j

Step 7 – To find the leaving vector

Min (-1, 2, 1) = -1. Hence s_1 is outgoing vector

Step 8 – To find the incoming vector

Max (Δ_1 / x_{11}, Δ_4 / x_{14}) = (-2/5, -1) = -2/5. So x_1 is incoming vector

Step 9 –The key element is -5/3. Proceed to next iteration

	$C_j \rightarrow$		-2	-1	0	0	0	
Basic variables	C_B	X_B	X_1	X_2	S_1	S_2	S_3	
x1	-2	3/5	1	0	-3/5	1/5	0	
x2	-1	6/5	0	1	4/5	-3/5	0	
s3	0	0	0	0	1	-1	1	
	Z- = ′ 12/5		0	0	2/5	1/5	0	←Δ_j

Step 10 – $\Delta_j \geq 0$ and $X_B \geq 0$, therefore the optimal solution is Max - =

x_1=3/5, x_2 = 6/5

$-x_1 - x_2 \leq -1$

$-2x_1 - 3x_2 \leq -2$

$x_1, x_2 \geq 0$

Example 2

Minimize Z = $3x_1 + x_2$

Subject to

$x_1 + x_2 \geq 1$

$2x_1 + 3x_2 \geq 2$

and $x_1 \geq 0$, $x_2 \geq 0$

Solution

Maximize Z =- 3 ′$x_1 - x_2$

Subject to

′, Z = 12/5

, and SLPP

Maximize Z =-3 ′$x_1 - x_2$

Subject to

$-x_1 - x_2 + s_1 = -1$

$-2x_1 - 3x_2 + s_2 = -2$

$x_1, x_2, s_1, s_2 \geq 0$

	$C_j \rightarrow$		-3	-1	0	0	
Basic variables	C_B	X_B	X_1	X_2	S_1	S_2	
s_1	0	-1	-1	-1	1	0	
s_2	0	-2	-2	-3	0	1	→
	Z0 = ′		3	↑ 1	0	0	←Δ_j
s_1	0	-1/3	-1/3	0	1	-1/3	→
x_2	-1	2/3	2/3	1	0	-1/3	
	Z2/3- = ′		7/3	0	0	↑ 1/3	←Δ_j
s_2	0	1	1	0	-3	1	
x_2	-1	1	1	1	-1	0	
	Z1- = ′		2	0	1	0	←Δ_j

$\Delta_j \geq 0$ and $X_B \geq 0$, therefore the optimal solution is Max Z1- = ′, Z = 1, and $x_1 = 0$, $x_2 = 1$

Example 3

Max $Z = -2x_1 - x_3$

Subject to

$x_1 + x_2 - x_3 \geq 5$

$x_1 - 2x_2 + 4x_3 \geq 8$

and $x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$

Solution

Max $Z = -2x_1 - x_3$

Subject to

$-x_1 - x_2 + x_3 \leq -5$

$-x_1 + 2x_2 - 4x_3 \leq -8$

$x_1, x_2, x_3 \geq 0$

SLPP

Max $Z = -2x_1 - x_3$

Subject to

$-x_1 - x_2 + x_3 + s_1 = -5$

$-x_1 + 2x_2 - 4x_3 + s_2 = -8$

$x_1, x_2, x_3, s_1, s_2 \geq 0$

	$C_j \rightarrow$		-2	0	-1	0	0	
Basic variables	C_B	X_B	X_1	X_2	X_3	S_1	S_2	
s_1	0	-5	-1	-1	1	1	0	
s_2	0	-8	-1	2	-4	0	1	→
					↑			
	Z = 0		2	0	1	0	0	←Δ_j
s_1	0	-7	-5/4	-1/2	0	1	1/4	→
x_3	-1	2	1/4	-1/2	1	0	-1/4	
				↑				
	Z = -2		7/4	1/2	0	0	1/4	←Δ_j
x_2	0	14	5/2	1	0	-2	-1/2	
x_3	-1	9	3/2	0	1	-1	-1/2	
	Z = -9		1/2	0	0	1	1/2	←Δ_j

$\Delta_j \geq 0$ and $X_B \geq 0$, therefore the optimal solution is Z = -9, and $x_1 = 0$, $x_2 = 14$, $x_3 = 9$

Check Your Progress

1. What is simplex method?

2. Define simplex algorithm.

3. What is a two-phase simplex method?

4. What are the methods used to solve an LPP involving artificial variables?

Let us sum up

Simplex method is an iterative procedure for solving LPP in a finite number of steps. It provides an algorithm, which consists of moving from one vertex of the region of feasible solution to another in such a manner that the value of the objective function at the succeeding vertex is less or more, as the case may be, than at the previous vertex. The simplex algorithm is the classical method to solve the optimization problem of linear programming. There are two methods used to solve an LPP involving artificial variables, namely: (i) Big M method (ii) Two-phase simplex method . The two-phase simplex method is another method to solve a given LPP involving some artificial variables Every LPP (called the primal) is associated with another LPP (called its dual). Either of the problem can be considered as primal with the other as dual. In the formulation of a dual problem, it should be first converted into its canonical form. The number of constraints in the dual will be equal to the number of variables in its primal and vice versa. In a dual, if one is a minimization problem, then the other will be a maximization problem. The fundamental duality theorem states that in the LPP if the primal problem has a finite optimal solution, then its dual also has a finite optimal solution.

SELF ASSESSMENT QUESTIONS

1. A factory produces three using three types of ingredients viz. A, B and C in different proportions. The following table shows the requirements o various ingredients as inputs per kg of the products.

4	8	8
4	6	4
8	4	0

The three profits coefficients are 20, 20 and 30 respectively. The factory has 800 kg of ingredients A, 1800 kg of ingredients B and 500 kg of ingredient C. Determine the product mix which will maximize the profit and also find out maximum profit

2. Solve the following linear programming problem using two phase and M method.

   Maximize

   $12x_1 + 15x_2 + 9x_3$

   Subject to:

   $8x_1 + 16x_2 + 12x_3 \leq 250$

   $4x_1 + 8x_2 + 10x_3 \geq 80$

   $7x_1 + 9x_2 + 8x_3 = 105$

   $x_1, x_2, x_3 \geq 0$

3. Solve the following linear programming problem using simplex method.

   Maximize

   3x1 + 2x2

   Subject to:

   x1 -x2 ≤ 1 x1 + x2 ≥ 3 x1, x2 ≥ 0

4. A bed mart company is in the business of manufacturing beds and pillows. The company has 40 hours for assembly and 32 hours for finishing work per day. Manufacturing of a bed requires 4 hours forassembly and 2 hours in finishing. Similarly a pillow requires 2 hours for assembly and 4 hours for finishing. Profitability analysis indicates that every bed would contribute Rs.80, while a pillow contribution is Rs.55 respectively. Find out the daily production of the company to maximize the contribution (profit).

5. Maximize

   $1170x_1 + 1110x_2$

   Subject to:

   $9x_1 + 5x_2 \geq 500$

   $7x_1 + 9x_2 \geq 300$

$5x_1 + 3x_2 \leq 1500$
$7x_1 + 9x_2 \leq 1900$
$2x_1 + 4x_2 \leq 1000$
$x_1, x_2 \geq 0$
Find graphically the feasible region and the optimal solution.

Glossary

Basic Variable: Variable of a basic feasible solution has n non-negative value.

Non Basic Variable: Variable of a feasible solution has a value equal to zero.

Artificial Variable: A non-negative variable introduced to provide basic feasible solution and initiate the simplex procedures.

Slack Variable: A variable corresponding to a ≤ type constraint is a non-negative variable introduced to convert the inequalities into equations.

Surplus Variable: A variable corresponding to a ≥ type constraint is a non-negative variable introduced to convert the constraint into equations.

Basic Solution: System of m-equation and n-variables i.e. m<n is a solution where at least n-m variables are zero.

Basic Feasible Solution: System of m-equation and n-variables i.e. m<n is a solution where m variables are non-negative and n-m variables are zero.

Optimum Solution: A solution where the objective function is minimized or maximized.

Linear Programming: It refers to a mathematical technique that is used to determine the best possible outcome or solution from a given set of parameters or list of requirements, which are represented in the form of linear relationships.

Criterion function: It refers to an objective function which states the determinants of the quantity to be either maximized or minimized.

Linearity: It refers to the property of a mathematical relationship or function which means that it can be graphically represented as a straight line

Objective Function: is a linear function of the decision variables representing the objective of the manager/decision maker.

Constraints: are the linear equations or inequalities arising out of practical limitations.

Decision Variables: are some physical quantities whose values indicate the solution.

Feasible Solution: is a solution which satisfies all the constraints (including the non-negative) presents in the problem.

Feasible Region: is the collection of feasible solutions.

Multiple Solutions: are solutions each of which maximize or minimize the objective function.

Unbounded Solution: is a solution whose objective function is infinite.

Infeasible Solution: means no feasible solution

UNIT III

TRANSPORTATION PROBLEM AND ASSIGNMENT MODEL

Introduction to Transportation Problem

The Transportation problem is to transport various amounts of a single homogeneous commoditythat are initially stored at various origins, to different destinations in such a way that the total transportation cost is a minimum. It can also be defined as to ship goods from various origins to various destinations in such a manner that the transportation cost is a minimum. The availability as well as the requirements is finite. It is assumed that the cost of shipping is linear.

The transportation problem is a specific type of linear programming problem (LPP). These problems focus on the task of moving various quantities of a uniform product from multiple sources to different destinations, all while minimizing transportation costs. In this context, you will explore the applications of the transportation problem, as well as the methods and principles used to solve such challenges. Solutions to transportation problems are typically developed in two phases: the initial solution and the optimal solution. There are three primary methods to determine the initial solution: the North West Corner Rule, the Least Cost Method, and Vogel's Approximation Method (VAM). Among these, VAM is often favored because it tends to yield results that are very close to the optimal solution. An optimal solution represents a feasible outcome that minimizes the total transportation cost. Achieving this optimal state involves refining the initial solution through further analysis. The Modified Distribution (MODI) method is employed to find optimal solutions and conduct optimality checks.

Some Basic Definitions

- **Feasible Solution**

A set of non-negative individual allocations ($x_{ij} \geq 0$) which simultaneously removes deficiencies is called as feasible solution.

- **Basic Feasible Solution**

A feasible solution to 'm' origin, 'n' destination problem is said to be basic if the number of positive allocations are m+n-1. If the number of allocations is less than m+n-1 then it is called as **Degenerate Basic Feasible Solution**. Otherwise it is called as Non-Degenerate Basic Feasible Solution.

- **Optimum Solution**

A feasible solution is said to be optimal if it minimizes the total transportation cost.

North-West Corner Rule

Step 1

- The first assignment is made in the cell occupying the upper left-hand (north-west) cornerof the table.
- The maximum possible amount is allocated here i.e. x_{11} = min (a_1, b_1). This value of x_{11} isthen entered in the cell (1,1) of the transportation table.

Step 2

i. If $b_1 > a_1$, move vertically downwards to the second row and make the second allocationof amount x_{21} = min (a_2, $b_1 - x_{11}$) in the cell (2, 1).
ii. If $b_1 < a_1$, move horizontally right side to the second column and make the secondallocation of amount x_{12} = min ($a_1 - x_{11}$, b_2) in the cell (1, 2).
iii. If $b_1 = a_1$, there is tie for the second allocation. One can make a second allocation ofmagnitude x_{12} = min ($a_1 - a_1$, b_2) in the cell (1, 2) or x_{21} = min (a_2, $b_1 - b_1$) in the cell (2, 1)

Step 3

Start from the new north-west corner of the transportation table and repeat steps 1 and 2 until allthe requirements are satisfied.

Find the initial basic feasible solution by using North-West Corner Rule

1.

W→ F ↓	W_1	W_2	W_3	W_4	Factory Capacity
F_1	19	30	50	10	7
F_2	70	30	40	60	9
F_3	40	8	70	20	18
Warehouse Requirement	5	8	7	14	34

Solution

	W_1	W_2	W_3	W_5	Availability
F_1	5 (19)	2 (30)			7 2 0
F_2		6 (30)	3 (40)		9 3 0
F_3			4 (70)	14 (20)	18 14 0
Requirement	5 0	8 6 0	7 4 0	14 0	

Initial Basic Feasible Solution

$x_{11} = 5$, $x_{12} = 2$, $x_{22} = 6$, $x_{23} = 3$, $x_{33} = 4$, $x_{34} = 14$

The transportation cost is 5 (19) + 2 (30) + 6 (30) + 3 (40) + 4 (70) + 14 (20) = Rs. 1015

2.

1	5	3	3
3	3	1	2
0	2	2	3
2	7	2	4

Demand : 21 25 17 17

Supply : 34 15 12 19

Solution

	D_1	D_2	D_3	D_4	Supply
O_1	21 (1)	13 (5)			34 13 0
O_2		12 (3)	3 (1)		15 3 0
O_3			12 (2)		12 0
O_4			2 (2)	17 (4)	19 17
Demand	21 0	25 12 0	17 14 2 0	17 0	

Initial Basic Feasible Solution

$x_{11} = 21$, $x_{12} = 13$, $x_{22} = 12$, $x_{23} = 3$, $x_{33} = 12$, $x_{43} = 2$, $x_{44} = 17$

The transportation cost is 21 (1) + 13 (5) + 12 (3) + 3 (1) + 12 (2) + 2 (2) + 17 (4) = Rs. 221

3.

From	To					Supply
	2	11	10	3	7	4
	1	4	7	2	1	8
	3	1	4	8	12	9
Demand	3	3	4	5	6	

Solution

From	To					Supply
	3 (2)	1 (11)				4 1 0
		2 (4)	4 (7)	2 (2)		8 6 2 0
				3 (8)	6 (12)	

Initial Basic Feasible Solution

$x_{11} = 3, x_{12} = 1, x_{22} = 2, x_{23} = 4, x_{24} = 2, x_{34} = 3, x_{35} = 6$

The transportation cost is 3 (2) + 1 (11) + 2 (4) + 4 (7) + 2 (2) + 3 (8) + 6 (12) = Rs. 153

Lowest Cost Entry Method (Matrix Minima Method)

Step 1

Determine the smallest cost in the cost matrix of the transportation table. Allocate x_{ij} = min (a_i, b_j) in the cell (i, j)

Step 2

- If $x_{ij} = a_i$, cross out the i^{th} row of the table and decrease b_j by a_i. Go to step 3.
- If $x_{ij} = b_j$, cross out the j^{th} column of the table and decrease a_i by b_j. Go to step 3.
- If $x_{ij} = a_i = b_j$, cross out the i^{th} row or j^{th} column but not both.

Step 3

Repeat steps 1 and 2 for the resulting reduced transportation table until all the requirements are satisfied. Whenever the minimum cost is not unique, make an arbitrary choice among the minima.

Find the initial basic feasible solution using Matrix Minima method

1.

W_1	W_2	W_3	W_4	
19	30	50	10	
70	30	40	60	Availability 7 9 18
40	8	70	20	

Requirement 5 18 7 14

Solution

	W_1	W_2	W_3	W_4	
F_1	(19)	(30)	(50)	(10)	7
F_2	(70)	(30)	(40)	(60)	9
F_3	(40)	8 (8)	(70)	(20)	10
	5	X	7	14	

	W_1	W_2	W_3	W_4	
F_1	(19)	(30)	(50)	7 (10)	X
F_2	(70)	(30)	(40)	(60)	9
F_3	(40)	8 (8)	(70)	(20)	10
	5	X	7	7	

	W_1	W_2	W_3	W_4	
F_1	(19)	(30)	(50)	7 (10)	X
F_2	(70)	(30)	(40)	(60)	9
F_3	(40)	8 (8)	(70)	7 (20)	3
	5	X	7	X	

	W_1	W_2	W_3	W_4	
F_1	(19)	(30)	(50)	7 (10)	X
F_2					9
F_3	3 (40)	8 (8)	(70)	7 (20)	X
	2	X	7	X	

	W_1	W_2	W_3	W_4	
F_1	(19)	(30)	(50)	7 (10)	X
F_2	2 (70)	(30)	7 (40)	(60)	X
F_3	3 (40)	8 (8)	(70)	7 (20)	X
	X	X	X	X	

Initial Basic Feasible Solution

$x_{14} = 7$, $x_{21} = 2$, $x_{23} = 7$, $x_{31} = 3$, $x_{32} = 8$, $x_{34} = 7$

The transportation cost is 7 (10) + 2 (70) + 7 (40) + 3 (40) + 8 (8) + 7 (20) = Rs. 814

3.

	To					Availability
From	2	11	10	3	7	4
	1	4	7	2	1	8
	3	9	4	8	12	9
Requirement	3	3	4	5	6	

Solution

To

From				4 (3)		4 0
	3 (1)				5 (1)	8 5 0
		3 (9)	4 (4)	1 (8)	1 (12)	9 5 4 1 0
	3	3	4	5	6	
	0	0	0	1	1	
				0	0	

Initial Basic Feasible Solution

$x_{14} = 4$, $x_{21} = 3$, $x_{25} = 5$, $x_{32} = 3$, $x_{33} = 4$, $x_{34} = 1$, $x_{35} = 1$

The transportation cost is 4 (3) + 3 (1) + 5(1) + 3 (9) + 4 (4) + 1 (8) + 1 (12) = Rs. 78

Vogel's Approximation Method (Unit Cost Penalty Method)

Step1

For each row of the table, identify the **smallest** and the **next to smallest cost**. Determine the difference between them for each row. These are called **penalties**. Put them aside by enclosing them in the parenthesis against the respective rows. Similarly compute penalties for each column.

Step 2

Identify the row or column with the largest penalty. If a tie occurs then use an arbitrary choice. Let the largest penalty corresponding to the i^{th} row have the cost c_{ij}. Allocate the largest possible amount x_{ij} = min (a_i, b_j) in the cell (i, j) and cross out either i^{th} row or j^{th} column in the usual manner.

Step 3

Again compute the row and column penalties for the reduced table and then go to step 2. Repeat the procedure until all the requirements are satisfied.

Find the initial basic feasible solution using vogel's approximation method

1.

	W_1	W_2	W_3	W_4	Availability
F_1	19	30	50	10	7
F_2	70	30	40	60	9
F_3	40	8	70	20	18
Requirement	5	8	7	14	

Solution

	W_1	W_2	W_3	W_4	Availability	Penalty
F_1	19	30	50	10	7	19-10=9
F_2	70	30	40	60	9	40-30=10
F_3	40	8	70	20	18	20-8=12
Requirement	5	8	7	14		
Penalty	40-19=21	30-8=22	50-40=10	20-10=10		

	W_1	W_2	W_3	W_4	Availability	Penalty
F_1	(19)	(30)	(50)	(10)	7	9
F_2	(70)	(30)	(40)	(60)	9	10
F_3	(40)	**8**(8)	(70)	(20)	18/10	12
Requirement	5	8/0	7	14		
Penalty	21	22	10	10		

	W_1	W_2	W_3	W_4	Availability	Penalty
F_1	**5**(19)	(30)	(50)	(10)	7/2	9
F_2	(70)	(30)	(40)	(60)	9	20
F_3	(40)	**8**(8)	(70)	(20)	18/10	20
Requirement	5/0	X	7	14		
Penalty	21	X	10	10		

	W_1	W_2	W_3	W_4	Availability	Penalty
F_1	**5**(19)	(30)	(50)	(10)	7/2	40
F_2	(70)	(30)	(40)	(60)	9	20
F_3	(40)	**8**(8)	(70)	**10**(20)	18/10/0	50
Requirement t	X	X	7	14/4		
Penalty	X	X	10	10		

	W_1	W_2	W_3	W_4	Availability	Penalty
F_1	**5**(19)	(30)	(50)	**2**(10)	7/2/0	40
F_2	(70)	(30)	(40)	(60)	9	20
F_3	(40)	**8**(8)	(70)	**10**(20)	X	X
Requirement	X	X	7	14/4/2		
Penalty	X	X	10	50		

	W_1	W_2	W_3	W_4	Availability	Penalty
F_1	**5**(19)	(30)	(50)	**2**(10)	X	X
F_2	(70)	(30)	**7**(40)	**2**(60)	X	X
F_3	(40)	**8**(8)	(70)	**10**(20)	X	X
Requirement	X	X	X	X		
Penalty	X	X	X	X		

Initial Basic Feasible Solution

$x_{11} = 5$, $x_{14} = 2$, $x_{23} = 7$, $x_{24} = 2$, $x_{32} = 8$, $x_{34} = 10$

The transportation cost is 5 (19) + 2 (10) + 7 (40) + 2 (60) + 8 (8) + 10 (20) = Rs. 779

Transportation Algorithm for Minimization Problem (MODI Method)

Step 1

Construct the transportation table entering the origin capacities a_i, the destination requirement b_j and the cost c_{ij}

Step 2

Find an initial basic feasible solution by vogel's method or by any of the given method.

Step 3

For all the basic variables x_{ij}, solve the system of equations $u_i + v_j = c_{ij}$, for all i, j for which cell (i, j) is in the basis, starting initially with some $u_i = 0$, calculate the values of u_i and v_j on the transportation table

Step 4

Compute the cost differences $d_{ij} = c_{ij} – (u_i + v_j)$ for all the non-basic cells

Step 5

Apply optimality test by examining the sign of each d_{ij}

- If all $d_{ij} \geq 0$, the current basic feasible solution is optimal
- If at least one $d_{ij} < 0$, select the variable x_{rs} (most negative) to enter the basis.
- Solution under test is not optimal if any d_{ij} is negative and further improvement is required by repeating the above process.

Step 6

Let the variable x_{rs} enter the basis. Allocate an unknown quantity Ɵ to the cell (r, s). Then construct a loop that starts and ends at the cell (r, s) and connects some of the basic cells. The amount Ɵ is added to and subtracted from the transition cells of the loop in such a manner thatthe availabilities and requirements remain satisfied.

Step 7

Assign the largest possible value to the Ɵ in such a way that the value of at least one basic variable becomes zero and the other basic variables remain non-negative. The basic cell whose allocation has been made zero will leave the basis.

Step 8

Now, return to step 3 and repeat the process until an optimal solution is obtained.

Example 1

Find an optimal solution

	W_1	W_2	W_3	W_4	Availability
F_1	19	30	50	10	7
F_2	70	30	40	60	9
F_3	40	8	70	20	18
Requirement	5	8	7	14	

Solution

1. **Applying vogel's approximation method for finding the initial basic feasible solution**

W_1	W_2	W_3	W_4
5(19)	(30)	(50)	**2**(10)
(70)	(30)	**7**(40)	**2**(60)
(40)	**8**(8)	(70)	**10**(20)

Minimum transportation cost is 5 (19) + 2 (10) + 7 (40) + 2 (60) + 8 (8) + 10 (20) = Rs. 779

2. **Check for Non-degeneracy**

The initial basic feasible solution has m + n – 1 i.e. 3 + 4 – 1 = 6 allocations in independentpositions. Hence optimality test is satisfied.

3. **Calculation of u_i and v_j : - $u_i + v_j = c_{ij}$**

• (19)			• (10)
		• (40)	• (60)
	• (8)		• (20)
$v_1 = 29$	$v_2 = 8$	$v_3 = 0$	$v_4 = 20$

u_1= -10, u_2 = 40, u_3 = 0

Assign a 'u' value to zero. (Convenient rule is to select the u_i, which has the largest number of allocations in its row)
Let $u_3 = 0$, then
$u_3 + v_4$= 20 which implies 0 + v_4 = 20, so $v_4 = 20$
$u_2 + v_4$= 60 which implies u_2 + 20 = 60, so $u_2 = 40$
$u_1 + v_4$= 10 which implies u_1 + 20 = 10, so $u_1 = -10$
$u_2 + v_3$= 40 which implies 40 + v_3 = 40, so $v_3 = 0$
$u_3 + v_2$= 8 which implies 0 + v_2 = 8, so $v_2 = 8$
$u_1 + v_1$= 19 which implies -10 + v_1= 19, so $v_1 = 29$

4. Calculation of cost differences for non basic cells $d_{ij} = c_{ij} - (u_i + v_j)$

c_{ij}

•	(30)	(50)	•
(70)	(30)	•	•
(40)	•	(70)	•

$u_i + v_j$

•	-2	-10	•
69	48	•	•
29	•	0	•

$d_{ij} = c_{ij} - (u_i + v_j)$

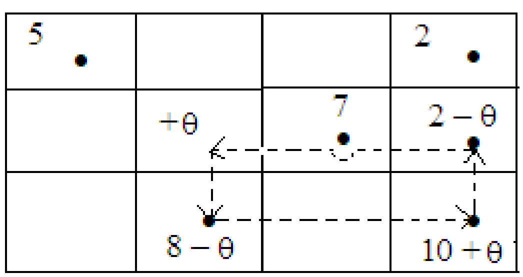
□	32	60	□
1	**-18**	□	□
11	□	70	□

5. Optimality test

$d_{ij} < 0$ i.e. $d_{22} = -18$

so x_{22} is entering the basis

6. Construction of loop and allocation of unknown quantity Ө



We allocate Ө to the cell (2, 2). Reallocation is done by transferring the maximum possible amount Ө in the marked cell. The value of Ө is obtained by equating to zero to the corners of the closed loop. i.e. min(8-Ө, 2-Ө) = 0 which gives Ө = 2. Therefore x_{24} is outgoing as it becomes zero.

5 (19)			2 (10)
	2 (30)	7 (40)	
	6 (8)		12 (20)

Minimum transportation cost is 5 (19) + 2 (10) + 2 (30) + 7 (40) + 6 (8) + 12 (20) = Rs. 743

7. Improved Solution

				u_i
• (19)			• (10)	$u_1 = -10$
	• (30)	• (40)		$u_2 = 22$
	• (8)		• (20)	$u_3 = 0$
$v_1 = 29$	$v_2 = 8$	$v_3 = 18$	$v_4 = 20$	v_j

c_{ij}

•	(30)	(50)	•
(70)	•	•	(60)
(40)	•	(70)	•

$u_i + v_j$

•	-2	8	•
51	•	•	42
29	•	18	•

$d_{ij} = c_{ij} – (u_i + v_j)$

□	32	42	□
19	□	□	18
11	□	52	□

Since $d_{ij} > 0$, an optimal solution is obtained with minimal cost Rs.743

Example 2

Solve by lowest cost entry method and obtain an optimal solution for the following problem

				Available
	50	30	220	1
	90	45	170	3
From	250	200	50	4
Required	4	2	2	

Solution

By lowest cost entry method

			Available
	1(30)		1/0
2(90)	1(45)		3/2/0
2(250)		2(50)	4/2/0
Required 4/2/2/0	2/1/0	2/0	

From

Minimum transportation cost is 1 (30) + 2 (90) + 1 (45) + 2 (250) + 2 (50) = Rs. 855

Check for Non-degeneracy

The initial basic feasible solution has m + n – 1 i.e. 3 + 3 – 1 = 5 allocations in independentpositions. Hence optimality test is satisfied.

Calculation of u_i and v_j : - $u_i + v_j = c_{ij}$

			u_i
	• (30)		u_1 = -15
• (90)	• (45)		u_2 = 0
• (250)		• (50)	u_3 = 160
v_1 = 90	v_2 = 45	v_3 = -110	

Calculation of cost differences for non-basic cells $d_{ij} = c_{ij} – (u_i + v_j)$

c_{ij}

50	•	220
•	•	170
•	200	•

$u_i + v_j$

75	•	-125
•	•	-110
•	205	•

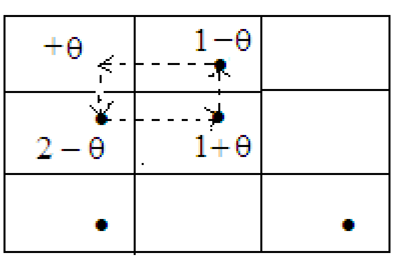
$d_{ij} = c_{ij} – (u_i + v_j)$

-25	□	345
□	□	280
□	-5	□

Optimality test

$d_{ij} < 0$ i.e. $d_{11} = -25$ is most negativeSo x_{11} is entering the basis

Construction of loop and allocation of unknown quantity ϴ



min(2-ϴ, 1-ϴ) = 0 which gives ϴ = 1. Therefore x_{12} is outgoing as it becomes zero.

1(50)		
1(90)	2(45)	
2(250)		2(50)

Minimum transportation cost is 1 (50) + 1 (90) + 2 (45) + 2 (250) + 2 (50) = Rs. 830

I Iteration

Calculation of u_i and v_j : - $u_i + v_j = c_{ij}$

				u_i
	• (50)			u_1= -40
	• (90)	• (45)		u_2 = 0
	• (250)		• (50)	u_3 = 160
v_j	v_1 = 90	v_2 = 45	v_3 = -110	

Calculation of $d_{ij} = c_{ij} – (u_i + v_j)$

c_{ij}

•	30	220
•	•	170
•	200	•

$u_i + v_j$

•	5	-150
•	•	-110
•	205	•

$d_{ij} = c_{ij} – (u_i + v_j)$

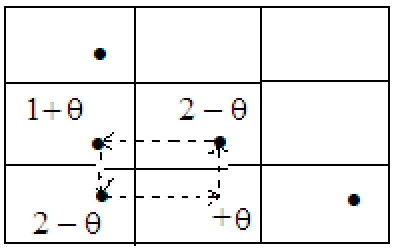
□	25	370
□	□	280
□	-5	□

Optimality test

$d_{ij} < 0$ i.e. $d_{32} = -5$

So x_{32} is entering the basis

Construction of loop and allocation of unknown quantity ϴ



2 – ϴ = 0 which gives ϴ = 2. Therefore x_{22} and x_{31} is outgoing as it becomes zero.

1(50)		
3(90)	0(45)	
	2(200)	2(50)

Minimum transportation cost is 1 (50) + 3 (90) + 2 (200) + 2 (50) = Rs. 820

II Iteration

Calculation of u_i and v_j : - $u_i + v_j = c_{ij}$

				u_i
	• (50)			u_1= -40
	• (90)	• (45)		u_2 = 0
		• (200)	• (50)	u_3 = 155
v_j	v_1 = 90	v_2 = 45	v_3 = -	

105

Calculation of $d_{ij} = c_{ij} – (u_i + v_j)$

c_{ij}

•	30	220
•	•	170
250	•	•

$u_i + v_j$

•	5	-145
•	•	-105
245	•	•

$d_{ij} = c_{ij} – (u_i + v_j)$

□	25	365
□	□	275
5	□	□

Since $d_{ij} > 0$, an optimal solution is obtained with minimal cost Rs.820

Check Your Progress

1. What is a transportation problem?
2. Define feasible, basic, non-degenerate solutions of a transportation problem.
3. List the approaches used with transportation problems for determining the starting solution.
4. State the optimal solution to a transportation problem.
5. What is the purpose of the MODI method?
6. State the two conditions necessary for an alternate solution.

Let us sum up

The transportation problem (TP) is one of the subclasses of LPP (Linear Programming Problem) in which the objective is to transport various quantities of a single homogeneous commodity that are initially stored at various origins to different destinations in such a way that the transportation cost is minimum. Any set of non-negative allocations ($X_{ij} > 0$) which satisfies the row and column sum (rim requirement) is called a feasible solution. A feasible solution is called a basic feasible solution if the number of nonnegative allocations is equal to $m + n – 1$, where m is the number of rows and n the number of columns in a transportation table. Any feasible solution to a transportation problem containing m origins and n destinations is said to be non-degenerate, if it contains $m + n – 1$ occupied cells and each allocation is in independent positions. The allocations are said to be in independent positions if it is impossible to form a closed path. Closed path means by allowing horizontal and vertical lines and when all the corner cells are occupied. If a basic feasible solution contains less than $m + n – 1$ non-negative allocations, it is said to be degenerate basic feasible solution. An optimal solution is a feasible solution (not necessarily basic) which minimizes the total cost. The solution of a transportation problem (TP) can be obtained in two stages, namely initial solution and optimum solution. The cells in the transportation table can be classified into occupied cells and unoccupied cells. The allocated cells in the transportation table are called occupied cells and the empty cells in the transportation table are called unoccupied cells. Optimality test can be conducted to any initial basic feasible solution of a TP, provided such allocations has exactly $m + n – 1$ non-negative allocations, where m is the number of origins and n is the number of destinations. Also, these allocations must be in independent positions. To perform this optimality test, modified distribution method (MODI) is used. MODI method is applied in order to determine the optimum solution. One can determine a set of numbers u_i and v_j for each row and column, with $u_i + v_j = C_{ij}$ for each occupied cell. To start with, we give $u_2 = 0$ as the second row has the maximum number of allocation.

SELF ASSESSMENT QUESTIONS AND EXERCISES

Short Answer Questions

1. What are the numbers of non-basic variables for 4 rows and 5 columns?
2. While dealing with North West Corner rule, when does one move to the next cell in next column?

3. What is the coefficient of Xij of constraints in a transportation problem?

4. When does degeneracy occur in an m × n transportation problem?

Long Answer Questions

1. What do you understand by transportation model?

2. Explain the following with examples: (i) North West Corner Rule (ii) Least Cost Method (iii) Vogel's Approximation Method

3. Explain degeneracy in a transportation problem. Describe a method to resolve it.

4. What do you mean by an unbalanced transportation problem?

5. Explain the process of converting an unbalanced transportation problem into a balanced one.

6. Obtain the optimal cost to the following transportation problem. Use least cost method (row/ matrix minima method) to get initial cost.

	A	B	C	D	Supply
I	5	2	4	3	22
II	4	8	1	6	15
III	4	6	7	5	8
Demand	7	12	17	9	

7.Find the optimal cost to the following transportation problem using VAM to get initial solution

	D1	D2	D3	D4	supply
O1	12	18	13	20	50
O2	17	11	16	15	60
O3	11	10	14	13	40
Demand	20	25	10	35	

Introduction to Assignment Problem

In assignment problems, the objective is to assign a number of jobs to the equal number ofpersons at a minimum cost of maximum profit.

Suppose there are 'n' jobs to be performed and 'n' persons are available for doing these jobs. Assume each person can do each job at a time with a varying degree of efficiency. Let c_{ij} be the cost of i^{th} person assigned to j^{th} job. Then the problem is to find an assignment so that the total cost for performing all jobs is minimum. Such problems are known as **assignment problems.**

These problems may consist of assigning men to offices, classes to the rooms or problems to theresearch team etc.

Algorithm for Assignment Problem (Hungarian Method)

Step 1

Subtract the minimum of each row of the effectiveness matrix, from all the elements of therespective rows (Row reduced matrix).

Step 2

Further modify the resulting matrix by subtracting the minimum element of each column from allthe elements of the respective columns. Thus first modified matrix is obtained.

Step 3

Draw the minimum number of horizontal and vertical lines to cover all the zeroes in the resultingmatrix. Let the minimum number of lines be N. Now there may be two possibilities

- If N = n, the number of rows (columns) of the given matrix then an optimal assignmentcan be made. So make the zero assignment to get the required solution.
- If N < n then proceed to step 4

Step 4

Determine the smallest element in the matrix, not covered by N lines. Subtract this minimum element from all uncovered elements and add the same element at the intersection of horizontal and vertical lines. Thus the second modified matrix is obtained.

Step 5

Repeat step 3 and step 4 until minimum number of lines become equal to number of rows (columns) of the given matrix i.e. N = n.

Step 6

To make zero assignment - examine the rows successively until a row-wise exactly single zero isfound; mark this zero by ' 'to make the assignment. Then, mark a 'X' over all zeroes if lying in the column of the marked zero, showing that they cannot be considered for further assignment. Continue in this manner until all the rows have been examined. Repeat the same procedure for the columns also.

Step 7

Repeat the step 6 successively until one of the following situations arise

- If no unmarked zero is left, then process ends
- If there lies more than one of the unmarked zeroes in any column or row, then mark' 'one of the unmarked zeroes arbitrarily and mark a cross in the cells of remaining zeroes in its row and column. Repeat the process until no unmarked zero is left in the matrix.

Step 8

Exactly one marked zero in each row and each column of the matrix is obtained. The assignmentcorresponding to these marked zeroes will give the optimal assignment.

Worked Examples

Example 1

A department head has four subordinates and four tasks have to be performed. Subordinates differ in efficiency and tasks differ in their intrinsic difficulty. Time each man would take to perform each task is given in the effectiveness matrix. How the tasks should be allocated to each person so as to minimize the total man-hours?

Subordinates

Tasks	I	II	III	IV
A	8	26	17	11
B	13	28	4	26
C	38	19	18	15
D	19	26	24	10

Solution

Row Reduced Matrix

0	18	9	3
9	24	0	22
23	4	3	0
9	16	14	0

I Modified Matrix

0	14	9	3
9	20	0	22
23	0	3	0
9	12	14	0

N = 4, n = 4

Since N = n, we move on to zero assignment

Optimal assignment	A – I	B – III	C – II	D – IV
Man-hours	8	4	19	10

0	14	9	3
9	20	0	22
23	0	3	0
9	12	14	0

Total man-hours = 8 + 4 + 19 + 10 = 41 hours

Example 2

A car hire company has one car at each of five depots a, b, c, d and e. a customer requires a car in each town namely A, B, C, D and E. Distance (kms) between depots (origins) and towns (destinations) are given in the following distance matrix

	a	b	c	d	e
A	160	130	175	190	200
B	135	120	130	160	175
C	140	110	155	170	185
D	50	50	80	80	110
E	55	35	70	80	105

Solution

Row Reduced Matrix

30	0	45	60	70
15	0	10	40	55
30	0	45	60	75
0	0	30	30	60
20	0	35	45	70

I Modified Matrix

30	0	35	30	15
15	0	0	10	0
30	0	35	30	20
0	0	20	0	5
20	0	25	15	15

N < n i.e. 3 < 5, so move to next modified matrixII Modified Matrix

15	0	20	15	0
15	15	0	10	0
15	0	20	15	5
0	15	20	0	5
5	0	10	0	0

N = 5, n = 5

Since N = n, we move on to zero

assignmentZero assignment

15	0	20	15	[0]
15	15	[0]	10	0
15	[0]	20	15	5
[0]	15	20	0	5
5	0	10	[0]	0

Route	A – e	B – c	C – b	D – a	E – d
Distance	200	130	110	50	80

Minimum distance travelled = 200 + 130 + 110 + 50 + 80 = 570 kms

Example 3

Solve the assignment problem whose effectiveness matrix is given in the table

	1	2	3	4
A	49	60	45	61
B	55	63	45	69
C	52	62	49	68
D	55	64	48	66

Solution

Row-Reduced Matrix

4	15	0	16
10	18	0	24
3	13	0	19
7	16	0	18

I Modified Matrix

1	2	0	0
7	5	0	8
0	0	0	3
4	3	0	2

N < n i.e 3 < 4, so II modified matrix

II Modified Matrix

1	2	2	0
5	3	0	6
0	0	2	3
2	1	0	0

N < n i.e 3 < 4

III Modified matrix

0	1	2	0
4	2	0	6
0	0	3	4
1	0	0	0

Since N = n, we move on to zero assignment
Multiple optimal assignments exists

Solution - I

[0]	1	2	~~0~~
4	2	[0]	6
~~0~~	[0]	3	4
1	~~0~~	~~0~~	[0]

Optimal assignment	A – 1	B – 3	C – 2	D - 4
Value	49	45	62	66

Total cost = 49 + 45 + 62 + 66 = 222 units

Solution – II

~~0~~	1	2	[0]
4	2	[0]	6
[0]	~~0~~	3	4
1	[0]	~~0~~	~~0~~

Optimal assignment	A – 4	B – 3	C – 1	D – 2
Value	61	45	52	64

Minimum cost = 61 + 45 + 52 + 64 = 222 units

Example 4

Certain equipment needs 5 repair jobs which have to be assigned to 5 machines. The estimated time (in hours) that a mechanic requires to complete the repair job is given in the table. Assuming that each mechanic can be assigned only one job, determine the minimum time assignment.

	J1	J2	J3	J4	J5
M1	7	5	9	8	11
M2	9	12	7	11	10
M3	8	5	4	6	9
M4	7	3	6	9	5
M5	4	6	7	5	11

Solution

Row Reduced Matrix

2	0	4	3	6
2	5	0	4	3
4	1	0	2	5
4	0	3	6	2
0	2	3	1	7

I Modified Matrix

2	0	4	2	4
2	5	0	3	1
4	1	0	1	3
4	0	3	5	0
0	2	3	0	5

N < n

II Modified Matrix

1	0	4	1	3
1	5	0	2	0
3	1	0	0	2
4	1	4	5	0
0	3	4	0	5

N = n

Zero assignment

1	[0]	4	1	3
1	5	[0]	2	X
3	1	X	[0]	2
4	1	4	5	[0]
[0]	3	4	X	5

Optimal assignment	M1 – J2	M2 – J3	M3 – J4	M4 – J5	M5 – J1
Hours	5	7	6	5	4

Minimum time = 5 + 7 + 6 + 5 + 4 = 27 hours

Maximal Assignment Problem

Example 1

A company has 5 jobs to be done. The following matrix shows the return in terms of rupees on assigning i^{th} (i = 1, 2, 3, 4, 5) machine to the j^{th} job (j = A, B, C, D, E). Assign the five jobs to the five machines so as to maximize the total expected profit.

		Jobs				
		A	B	C	D	E
Machines	1	5	11	10	12	4
	2	2	4	6	3	5
	3	3	12	5	14	6
	4	6	14	4	11	7
	5	7	9	8	12	5

Solution

Subtract all the elements from the highest elementHighest element = 14

9	3	4	2	10
12	10	8	11	9
11	2	9	0	8
8	0	10	3	7
7	5	6	2	9

Row Reduced matrix

7	1	2	0	8
4	2	0	3	1
11	2	9	0	8
8	0	10	3	7
5	3	4	0	7

I Modified Matrix

3	1	2	0	7
0	2	0	3	0
7	2	9	0	7
4	0	10	3	6
1	3	4	0	6

$N < n$ i.e. $3 < 5$

II Modified Matrix

2	0	1	0	6
0	2	0	4	0
6	1	8	0	6
4	0	10	4	6
0	2	3	0	5

N < n i.e. 4 < 5

III Modified Matrix

1	0	0	0	5
0	3	0	5	0
5	1	7	0	5
3	0	9	4	5
0	3	3	1	5

N = n

Zero assignment

1	0	0	0	5
0	3	0	5	0
5	1	7	0	5
3	0	9	4	5
0	3	3	1	5

Optimal assignment 1 – C 2 – E 3 – D 4 – B 5 – AMaximum profit = 10 + 5 + 14 + 14 + 7 = Rs. 50

unbalanced Assignment Problem

If number of rows is not equal to number of columns then it is called Unbalanced Assignment Problem.

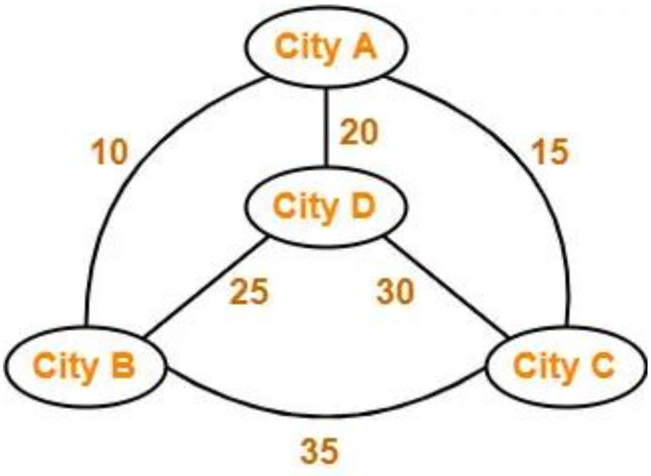
So to solve this problem, we have to add dummy rows or columns with cost 0, to make it a square matrix.

Excercise

Find Solution of Assignment problem using Hungarian method (MIN case)

Work\Job	I	II	III	IV
A	9	14	19	15
B	7	17	20	19
C	9	18	21	18
D	10	12	18	19
E	10	15	21	16

Travelling Salesman Problem



Travelling Salesman Problem

Rule

A travelling salesman plans to visit n cities. He wishes to visit each city only once, and again arriving back to his home city from where he started. So that the total travelling distance is minimum.

If there are n cities, then there are (n - 1)! possible ways for his tour. For example, if the number of cities to be visited is 4, then there are 3! = 6 different combination is possible. Such type of problems can be solved by Hungarian method, branch and bound method, penalty method, nearest neighbour method.

Example

Solve the TSP given by C12 = 20, C13 = 4, C14 = 10, C23 = 5, C32=6, C25=10, C35=6, C45=20

Where Cij = Cji, there is no routes between i and j, if the value cij is not shown

Table:1

	1	2	3	4	5
1	∞	20	4	10	∞
2	20	∞	5	∞	10
3	4	5	∞	6	6
4	10	∞	6	∞	20
5	∞	10	6	20	∞

Table:2

	1	2	3	4	5
1	∞	20	4	10	∞
2	20	∞	5	∞	10
3	4	5	∞	6	6
4	10	∞	6	∞	20

5	∞	10	6	20	∞

Table:3

	1	**2**	**3**	**4**	**5**
1	∞	15	0	4	∞
2	15	∞	0	∞	3
3	0	0	∞	0	0
4	4	∞	0	∞	12
5	∞	3	0	12	∞

Table: 4

	1	**2**	**3**	**4**	**5**
1	∞	12	0	1	∞
2	12	∞	0	∞	0
3	0	0	∞	0	0
4	1	∞	0	∞	9
5	∞	0	0	9	∞

After Column ReductionTable: 5

	1	**2**	**3**	**4**	**5**
1	∞	12	0	0	∞
2	11	∞	0	∞	0
3	0	1	∞	0	1
4	1	∞	0	∞	9
5	∞	0	0	8	∞

Table: 6

	1	**2**	**3**	**4**	**5**
1	∞	11	0	0	∞
2	12	∞	1	∞	0
3	0	0	∞	0	0
4	0	∞	0	∞	8
5	∞	0	0	8	∞

	1	**2**	**3**	**4**	**5**
1	∞	20	4	10	∞
2	20	∞	5	∞	10
3	4	5	∞	6	6
4	10	∞	6	∞	20
5	∞	10	6	20	∞

TSP Cost = 4+10+5+10+20 = 49 /-

Cycle: 1-3-2-5-4-1

Exercise:

1. A classic example of the Traveling Salesman Problem (TSP) is that of a delivery driver who needs to visit a set of customers in different locations, making only one stop at each customer's location and returning to the starting point. The goal is to find the shortest possible route that accomplishes this.

Example: Imagine a delivery driver needs to visit four customers, A, B, C, and D, located in different parts of a city. The distances between these locations are:

- A to B: 5 miles
- A to C: 8 miles
- A to D: 7 miles
- B to C: 6 miles
- B to D: 3 miles
- C to D: 4 miles

This information can be represented as a distance matrix:

	A	B	C	D
A	0	5	8	7
B	5	0	6	3
C	8	6	0	4
D	7	3	4	0

2. Find Solution of Travelling salesman problem (MIN case)

Work\Job	1	2	3	4
1	x	4	9	5
2	6	x	4	8
3	9	4	x	9
4	5	8	9	x

Check Your Progress

1. State any one difference between transportation and assignment problems.
2. What is an unbalanced assignment problem?
3. What do you mean by routing problem?

Let us sum up

An assignment problem is one of the fundamental combinatorial optimization problems and helps to find a maximum weight identical in nature in a weighted bipartite graph. The solution of an assignment problem can be arrived at using the Hungarian method. An assignment problem is balanced if the cost matrix is a square matrix; otherwise, it is termed as unbalanced. To convert an unbalanced assignment problem into a balanced problem, dummy rows or columns are added with all entries as 0s.

SELF ASSESSMENT QUESTIONS

Short Answer Questions

1. State the differences between a transportation problem and an assignment problem.

2. Give a mathematical formulation of the assignment problem.

3. How can you maximize an objective function in the assignment problem?

Long Answer Questions

1. Discuss the assignment problem with a suitable example.

2. Describe the algorithm for the solution of the assignment problem.

3. Explain the nature of i in travelling salesman problem and give its mathematical formulation.

4. Solve the following assignment problem**.**

	I	II	III	IV	V
1	10	5	13	15	16
2	3	9	18	3	6
3	10	7	2	2	2
4	5	11	9	7	12
5	7	9	10	4	12

Multiple Choice Questions

1. The degeneracy in the transportation problem indicates that

(a) Dummy allocation needs to be added
(b) The problem has no feasible solution
(c) The multiple optimal solution exists.
(d) (a) and (b) only

2. When the total supply is not equal to total demand in a transportation problem then it iscalled

(a) Balanced
(b) Unbalanced
(e) Degenerate
(d) None of these

3. The solution to a transportation problem with m-rows and n-columns is feasible if numberof positive allocations are

(a) m + n
(b) m *n
(c) m+n-l
(d) m+n+l

4. An assignment problem is considered as a particular case of a transportation problembecause
   A. The number of rows equals columns
   B. All xij= 0 or 1
   C. All rim conditions are 1
   D. All of the above

5. An optimal assignment requires that the maximum number of lines that can be drawnthrough squares with zero opportunity cost be equal to the number of
   A. Rows or columns
   B. Rows & columns
   C. Rows + columns –1 d.
   D. None of the above

6. While solving an assignment problem, an activity is assigned to a resource through asquare with zero opportunity cost because the objective is to
   A. Minimize total cost of assignment
   B. Reduce the cost of assignment to zero
   C. Reduce the cost of that particular assignment to zero
   D. All of the above

7. The method used for solving an assignment problem is called
   A. Reduced matrix method
   B. MODI method
   C. Hungarian method
   D. None of the above

8. The purpose of a dummy row or column in an assignment problem is to
   A. Obtain balance between total activities &total resources
   B. Prevent a solution from becoming degenerate
   C. Provide a means of representing a dummy problem
   D. None of the above

9. Maximization assignment problem is transformed into a minimization problem by
   A. Adding each entry in a column from the maximization value in that column
   B. Subtracting each entry in a column from the maximum value in that column
   C. Subtracting each entry in the table from the maximum value in that table
   D. Any one of the above

10. If there were n workers & n jobs there would be
   A. n! solutions
   B. (n-1)! solutions
   C. (n!)nsolutions
   D. n solutions

11. An assignment problem can be solved by
   A. Simplex method
   B. Transportation method
   C. Both a & b
   D. None of the above

12. For a salesman who has to visit n cities which of the following are the ways of his tourplan
   A. n!
   B. (n+1)!
   C. (n-1)!
   D. n

13. The assignment problem
   A. Requires that only one activity be assigned to each resource
   B. Is a special case of transportation problem
   C. Can be used to maximize resources
   D. All of the above

14. An assignment problem is a special case of transportation problem, where
   A. Number of rows equals number of columns
   B. All rim conditions are 1
   C. Values of each decision variable is either 0 or 1
   D. All of the above

15. Every basic feasible solution of a general assignment problem, having a square pay-offmatrix of order, n should have assignments equal t
   A. 2n+1
   B. 2n-1
   C. m+n-1

D. m+n

16. To proceed with the MODI algorithm for solving an assignment problem, the number ofdummy allocations need to be added are
    A. n
    B. 2n
    C. n-1
    D. 2n-1

17. The Hungarian method for solving an assignment problem can also be used to solve
    A. A transportation problem
    B. A travelling salesman problem
    C. A LP problem
    D. Both a & b

18. An optimal solution of an assignment problem can be obtained only if
    A. Each row & column has only one zero element
    B. Each row & column has at least one zero element
    C. The data is arrangement in a square matrix
    D. None of the above

19. Which method usually gives a very good solution to the assignment problem?
    A. northwest corner rule
    B. Vogel's approximation method
    C. MODI method
    D. stepping-stone method

20. In applying Vogel's approximation method to a profit maximization problem, row andcolumn penalties are determined by:
    A. finding the largest unit cost in each row or column.
    B. finding the smallest unit cost in each row or column.
    C. finding the sum of the unit costs in each row or column.
    D. finding the difference between the two lowest unit costs in each row and column.
    E. finding the difference between the two highest unit costs in each row and column.

21. The northwest corner rule requires that we start allocating units to shipping routes in the: middle cell.
    A. Lower right corner of the table.
    B. Upper right corner of the table.
    C. Highest costly cell of the table.
    D. Upper left-hand corner of the table.

.

22. The table represents a solution that is:

To=>	1	2	3	Supply
From A	3	6 40	4	40
B	3	4 30	5	30
C	5 20	7	6 10	30

A. an initial solution.
B. Infeasible.
C. degenerate.
D. all of the above
E. none of the above

23. Which of the following is used to come up with a solution to the assignment problem?
A. MODI method
B. northwest corner method
C. stepping-stone method
D. Hungarian method
E. none of the above

24. What is wrong with the following table?

	To==>	1	2	3	Dummy	Supply
From	A	10	8 80	12	0 20	100
	B	6 120	7 40	4 30	0	150
	C	10	9 10	6 170	0 80	250
	Demand	120	80	200	100	

A. **The solution is infeasible**.
B. The solution is degenerate.
C. The solution is unbounded.
D. Nothing is wrong.
E. The solution is inefficient in that it is possible to use fewer routes.

25. The solution presented in the following table is

Table 10-4						
	To==>	1	2	3	Dummy	Supply
From	A	10	8 80	12	0 20	100
	B	6 120	7	4 30	0	150
	C	10	9	6 170	0 80	250
	Demand	120	80	200	100	

A. infeasible.
B. degenerate.
C. unbounded.
D. Optimal.
E. none of the above

26. The solution shown was obtained by Vogel's approximation. The difference between theobjective function for this solution and that for the optimal is

	To==>	1	2	3	Dummy	Supply
From	A	10	8	12	0 100	100
	B	6 120	7	4 30	0	150
	C	10	9 80	6 170	0	250
	Demand	120	80	200	100	

A. 40
B. 60
C. 80
D. 100
E. none of the above

27. Optimal solution of an assignment problem can be obtained only if
A. Each row & column has only one zero element
B. Each row & column has at least one zero element
C. The data is arrangement in a square matrix
D. None of the above

28. In assignment problem of maximization, the objective is to maximise
A. Profit
B. optimization

C. cost
D. None of the above

29. What is the difference between minimal cost network flows and transportation problems?

A. The minimal cost network flows are special cases of transportation problems
B. The transportation problems are special cases of the minimal cost network flows
C. There is no difference
D. The transportation problems are formulated in terms of tableaus, while the minimalcost network flows are formulated in terms of graphs

30. With the transportation technique, the initial solution can be generated in any fashionone chooses. The only restriction is that
A. the edge constraints for supply and demand are satisfied.
B. the solution is not degenerate.
C. the solution must be optimal.
D. one must use the northwest-corner method.

31. The purpose of the stepping-stone method is to
A. develop the initial solution to the transportation problem.
B. assist one in moving from an initial feasible solution to the optimal solution.
C. determine whether a given solution is feasible or not.
D. identify the relevant costs in a transportation problem.

32. The purpose of a dummy source or dummy destination in a transportation problem isto
A. prevent the solution from becoming degenerate.
B. **obtain a balance between total supply and total demand**.
C. make certain that the total cost does not exceed some specified figure.
D. provide a means of representing a dummy problem.

33. Which of the following is NOT needed to use the transportation model?

A. the cost of shipping one unit from each origin to each destination
B. the destination points and the demand per period at each
C. the origin points and the capacity or supply per period at each
D. degeneracy

34. Which of the following is a method for improving an initial solution in a transportation problem?
A. northwest-corner
B. intuitive lowest-cost
C. southeast-corner rule
D. stepping-stone

35. The transportation method assumes that

A. **there are no economies of scale if large quantities are shipped from one source toone destination.**
B. the number of occupied squares in any solution must be equal to the number of rowsin the table plus the number of columns in the table plus 1.
C. there is only one optimal solution for each problem.
D. the number of dummy sources equals the number of dummy destinations.

36. An initial transportation solution appears in the table.

	C	D	Factory Capacity
A	10	0	10
B	15	25	40
Warehouse Demand	25	25	50

Can this solution be improved if it costs $5 per unit to ship from A to C; $7 per unit to shipfrom A to D; $8 to ship from B to C; and $9 to ship from B to D?

A. Yes, this solution can be improved by $50.
B. Yes, this solution can be improved by $100.
C. **No, this solution is optimal**.
D. Yes, the initial solution can be improved by $10.

37. What is the cost of the transportation solution shown in the table?

	W	X	Y	Supply
A	$3 20	$5 50	$9 0	70
B	$5 0	$4 30	$7 0	30
C	$10 40	$8 0	$3 80	120
Demand	60	80	80	220

A. $1350
B. $1070
C. $1150
D. $1230

38. Which statement regarding this transportation table is best?

	W	X	Y	Supply
A	$3 20	$5 50	$9 0	70
B	$5 0	$4 30	$7 0	30

C	$10 40	$8 0	$3 80	120
Deman d	60	80	80	220

A. The solution is degenerate.
B. This solution can be improved by shipping from C to X.
C. This solution would be improved by shipping from B to W.
D. This solution was developed using the northwest corner rule.

39. Which of these statements about the stepping-stone method is best?

A. A dummy source and destination must be added if the number of rows plus columnsminus 1 is not equal to the number of filled squares.
B. Only squares containing assigned shipments can be used to trace a path back toan empty square.
C. An improvement index that is a net positive means that the initial solution can beimproved.
D. Only empty squares can be used to trace a path back to a square containing anassigned shipment

40. In a transportation problem, we must make the number of ____________ __________equal.
A. destinations; sources
B. units supplied; units demanded
C. columns; rows
D. positive cost coefficients; negative cost coefficients
E. warehouses; suppliers

41. ______ or________are used to "balance" an assignment or transportationproblem.

A. Destinations; sources
B. Units supplied; units demanded
C. Dummy rows; dummy columns
D. Large cost coefficients; small cost coefficients
E. Artificial cells; degenerate cells

42. The net cost of shipping one unit on a route not used in the current transportationproblem solution is called the___.
A. change index
B. new index
C. MODI index
D. idle index
E. Improvement index

43. The procedure used to solve assignment problems wherein one reduces the originalassignment costs to a table of opportunity costs is called__.

A. stepping-stone method
B. matrix reduction
C. MODI method
D. northwest reduction
E. simplex reduction

44. The method of finding an initial solution based upon opportunity costs is called _________.

A. the northwest corner rule
B. Vogel's approximation
C. Johanson's theorem
D. Flood's technique
E. Hungarian method

45. An assignment problem can be viewed as a special case of transportation problem inwhich the capacity from each source is______and the demand at each destination is

A. 1; 1
B. Infinity; infinity
C. 0; 0
D. 1000; 1000
E. -1; -1

46. _____occurs when the number of occupied squares is less than the number of rows plus

A. Degeneracy
B. Infeasibility
C. Unboundedness
D. Unbalance
E. Redundancy

47. Both transportation and assignment problems are members of a category of LP problems called_______.

A. shipping problems
B. logistics problems
C. generalized flow problems
D. routing problems
E. network flow problems

48. The equation Ri + Kj = Cij is used to calculate____________.

A. an improvement index for the stepping-stone method
B. the opportunity costs for using a particular route
C. the MODI cost values (Ri, Kj)
D. the degeneracy index
E. optimality test

49. In case of an unbalanced problem, shipping cost coefficients of_______are assignedto each created dummy factory or warehouse.
A. very high positive costs
B. very high negative costs
C. 10
D. zero
E. one

50. The initial solution of a transportation problem can be obtained by applying any known method. However, the only condition is that
A. The solution be optimal
B. The rim conditions are satisfied
C. The solution not be degenerate
D. All of the above

51. The dummy source or destination in a transportation problem is added to
A. Satisfy rim conditions
B. Prevent solution from becoming degenerate
C. Ensure that total cost does not exceed a limit
D. None of the above

52. The occurrence of degeneracy while solving a transportation problem means that
A. Total supply equals total demand
B. The solution so obtained is not feasible
C. The few allocations become negative
D. None of the above

53. An alternative optimal solution to a minimization transportation problem exists whenever opportunity cost corresponding to unused route of transportation is:
A. Positive & greater than zero
B. Positive with at least one equal to zero
C. Negative with at least one equal to zero
D. None of the above

54. One disadvantage of using North-West Corner rule to find initial solution to the transportation problem is that
A. It is complicated to use

B. It does not take into account cost of transportation
C. It leads to a degenerate initial solution
D. All of the above

55. The solution to a transportation problem with 'm' rows (supplies) & 'n' columns (destination) is feasible if number of positive allocations are
A. m+n
B. m*n
C. m+n-1
D. m+n+1

56. If an opportunity cost value is used for an unused cell to test optimality, it should be
A. Equal to zero
B. Most negative number
C. Most positive number
D. Any value

57. During an iteration while moving from one solution to the next, degeneracy may occur when
A. The closed path indicates a diagonal move
B. Two or more occupied cells are on the closed path but neither of them represents acorner of the path.
C. Two or more occupied cells on the closed path with minus sign are tied for lowestcircled value
D. Either of the above

58. The large negative opportunity cost value in an unused cell in a transportation table ischosen to improve the current solution because
A. It represents per unit cost reduction
B. It represents per unit cost improvement
C. It ensure no rim requirement violation
D. None of the above

59. The smallest quantity is chosen at the corners of the closed path with negative sign tobe assigned at unused cell because
A. It improve the total cost
B. It does not disturb rim conditions
C. It ensure feasible solution
D. All of the above

60. When total supply is equal to total demand in a transportation problem, the problem issaid to be
A. Balanced

B. Unbalanced
C. Degenerate
D. None of the above

61. Which of the following methods is used to verify the optimality of the current solution of the transportation problem
    A. Least cost method
    B. Vogel's approximation method
    C. Modified distribution method
    D. All of the above

62. The degeneracy in the transportation problem indicates that
    A. Dummy allocation(s) needs to be added
    B. The problem has no feasible solution
    C. The multiple optimal solution exist
    D. a & b but not c

63. In a transportation problem, when the number of occupied routes is less than thenumber of rows plus the number of columns -1, we say that the solution is:
    A. Unbalanced.
    B. Infeasible.
    C. Optimal.
    D. impossible.
    E. Degenerate.

64. The only restriction we place on the initial solution of a transportation problem is that: we must have nonzero quantities in a majority of the boxes.
    A. all constraints must be satisfied.
    B. demand must equal supply.
    C. we must have a number (equal to the number of rows plus the number of columnsminus one) of boxes which contain nonzero quantities.
    D. None of the above

65. The initial solution of a transportation problem can be obtained by applying any known method. However, the only condition is that
    A. the solution be optimal
    B. the rim condition are satisfied
    C. the solution not be degenerate
    D. all of the above

66. The dummy source or destination in a transportation problem is added to
    A. satisfy rim condition
    B. prevent solution from becoming degenerate

C. ensure that total cost does not exceed a limit
D. all of the above

67. The occurrence of degeneracy while solving a transportation problem means that
    A. total supply equals total demand
    B. the solution so obtained is not feasible
    C. the few allocations become negative
    D. none of the above

68. An alternative optimal solution to a minimization transportation problem existswhenever opportunity cost corresponding to unused routes of transportation is:
    A. positive and greater than zero
    B. positive with at least one equal to zero
    C. negative with at least one equal to zero
    D. all of the above

69. One disadvantage of using North-West Corner Rule to find initial solution tothe transportation problem is that
    A. it is complicated to use
    B. it does not take into account cost of transportation
    C. it leads to degenerate initial solution
    D. all of the above

Glossary

Origin: is the location from which the shipments are dispatched.

Destination: is the location to which the shipments are transported.

Unit Transportation Cost: is the transportation cost per unit from an origin to destination

Hungarian Method: is a technique of solving assignment problems.

Assignment Problem: is a special kind of linear programming problem where the objective is to minimize the assignment cost or time.

Balanced Assignment Problem: is an assignment problem where the number of persons equal to the number of jobs.

Unbalanced Assignment Problem: is an assignment problem where the number of jobs is not equal to the number of persons.

Infeasible Assignment Problem: is an assignment problem where a particular person is unable to perform a particular job or certain job cannot be done by certain machines

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UNIT- IV

INTRODUCTION

Project Scheduling and Resource Management: Deterministic Inventory models – Purchasing & Manufacturing models – Probabilistic inventory models – Replacement model – Sequencing – Brief Introduction to Queuing models. Networking – Programme Evaluation and Review Technique (PERT) and Critical Path Method (CPM) for Project Scheduling- Crashing – Resource allocation and Resource Scheduling.

Unit Module Structuring

1. **Inventory models**
2. **Replacement Model**
3. **Sequencing**
4. **Networking**

4.1 INTRODUCTION:

INVENTORY CONTROL

Inventory is essential to provide flexibility in operating a system or organization. An inventory can be classified into raw materials inventory, work-in-process inventory and finished goods inventory. The raw material inventory removes dependency between suppliers and plants. The work-in-process inventory removes dependency between various machines of a product line. The finished goods inventory removes dependency between plants and its customers or market. The main functions of an inventory are: smoothing out irregularities in supply, minimizing the production cost and allowing organizations to cope up with perishable materials. Some important terminologies of inventory control are discussed now.

Inventory decisions: The following two basic inventory decisions are generally taken by managers.

1. When to replenish the inventory of an item?
2. How much of an item to order when the inventory of that item is to be replenished?

Costs of inventory systems: The following costs are associated with the inventory system.

1. Purchase price/unit
2. Ordering cost/order
3. Carrying cost/unit/period
4. Shortage cost/unit/period.

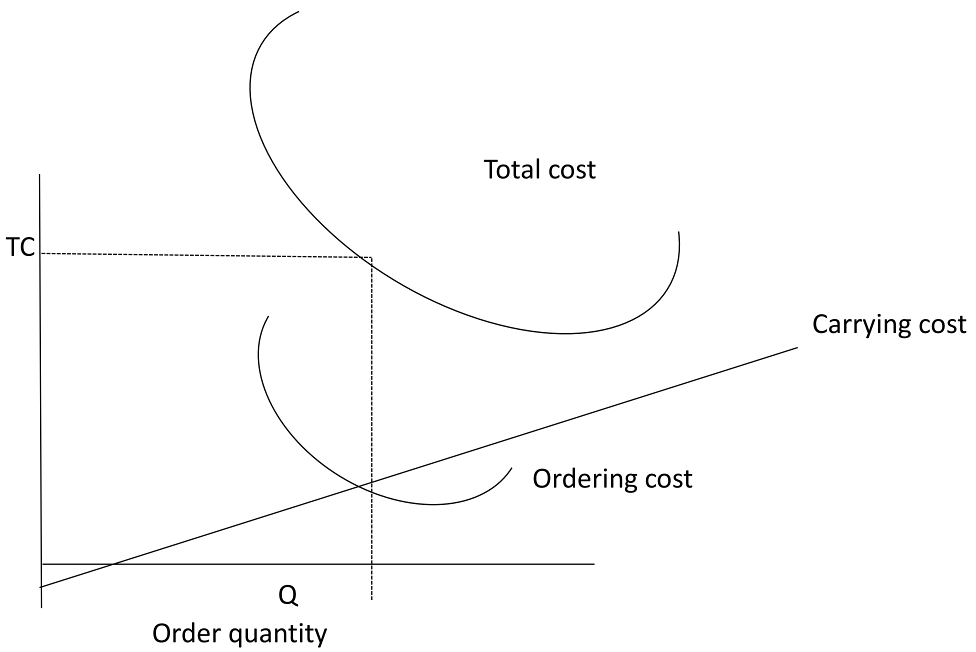
Costs trade off: If we place frequent orders, the cost of ordering will be more, but the inventory carrying cost will be less. On the other hand, if we place less frequent orders, the ordering cost will be less, but the carrying cost will be more. In the following figure, for an increase in Q (order size), the carrying cost increases and the ordering cost decreases. The total cost curve represents the sum of ordering cost and carrying cost for each order size. The order size at which the total cost is minimum is called economic order quantity (EOQ) or optimal order size (Q).

MODELS OF INVENTORY

There are different models of inventory. The inventory models can be classified into deterministic models and probabilistic models. The various deterministic models are:

(a) Purchase model with instantaneous replenishment and without shortages;
(b) Manufacturing model without shortages;
c) Purchase model with instantaneous replenishment and with shortages;
d) Manufacturing model with shortages

These models are explained in the following sections.



Purchase Model with Instantaneous Replenishment and without Shortages

In this inventory model, orders of equal size are placed at periodical intervals. The items against an order are replenished instantaneously and the items are consumed at a constant rate. The purchase price per unit is same irrespective of order size.

Let us suppose, D = Annual demand in units

C_0 = Ordering cost/order

C_c = Carrying cost/unit/year

P = Purchase price per unit

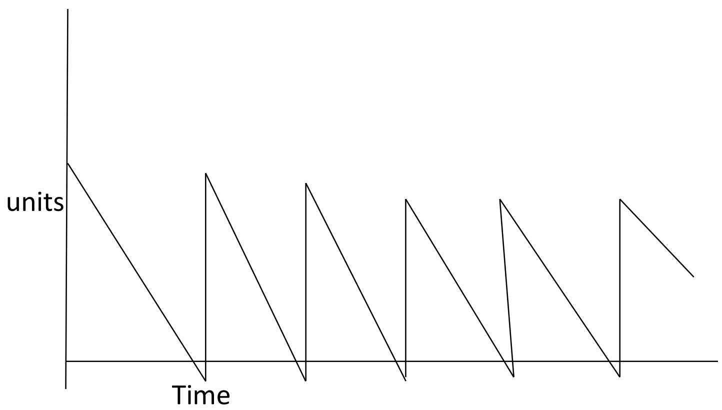
Q = Order size

The corresponding purchase model can be represented as shown in the following figure. From the above assumptions, we have:

The number of orders/year = $\frac{D}{Q}$

Average inventory = $\frac{Q}{2}$

Cost of ordering/year = $\frac{D}{Q}$ C_0



Cost of carrying/year = $\frac{Q}{2} C_C$

Therefore, Purchase cost/year = Dp

Total inventory cost/year = $\frac{D}{Q}$ C_0+$\frac{Q}{2}$ C_C+ Dp

Differentiating with respect to Q yields

$$\frac{d}{dQ}(TC) = \frac{-D}{Q^2}C_0 + \frac{C_c}{2}$$

Differentiating it again with respect to Q yields

$$\frac{d^2}{dQ^2}(TC) = \frac{2D}{Q^3}C_0$$

Since the second derivative is positive, the optimal value for Q is obtained by equating the first derivative to zero. Therefore,

$\frac{-D}{Q^2}C_0 + \frac{C_c}{2} = 0$ or $Q^2 = \frac{2C_o D}{C_c}$

Hence, the optimal order size is

Q = $\sqrt{\frac{2C_o D}{C_c}}$

and

Total number of orders per year = $\frac{D}{Q}$

Where

Time between orders = $\frac{Q}{D}$

Example:

Ram Industry needs 5400 units/year of a bought-out component which will be used in its main product. The ordering cost is Rs. 250 per order and the carrying cost per unit per year is Rs. 30. Find: the economic order quantity (EOQ), the number of orders per year and the time between successive orders.

Solution:

D = 5400 units/year

C_o = Rs. 250/order

C_c = Rs. 30/unit/year

Therefore, the economic order quantity

EOQ (Q*) = $\sqrt{\frac{2C_oD}{C_c}} = \sqrt{\frac{2\times250\times5400}{30}}$ =300 units

Where

Number of orders/year= $\frac{D}{Q} = \frac{5400}{300} = 18$

and

Time between successive orders = $\frac{Q}{D} = \frac{300}{5400}$ = 0.0556 year

= 0.6672 month

= 20 days (approx.)

Example: Alpha Industry needs 15,000 units per year of a bought-out component which will be used in its main product. The ordering cost is Rs. 125 per order and the carrying cost per unit per year is 20% of the purchase price per unit. The purchase price per unit is Rs. 75. Find: economic order quantity, number of orders per year and time between successive orders.

Solution:

We have

D = 15,000 units/year

Co = Rs.125/order

Purchase price/unit = Rs. 75

C_c = Rs. 75 x 0.20

= Rs. 15/unit/year

Therefore, the economic order quantity is

2CD Co 2 x 125 x 15,000 15 EOQ = $\sqrt{\frac{2\text{CoD}}{C_c}} = \sqrt{\frac{2\times125\times15000}{15}} =$ 500units

and

Number of orders/year = $\frac{D}{Q} = \frac{15000}{500}$ = 30

Time between successive orders is obtained as $\frac{Q}{D}$ = 5$\frac{500}{15,000}$ = 0.033 year

= 0.4 month

= 12 days

Manufacturing model without shortages

If a company manufactures an item which is required for its main product, then the corresponding model of inventory is called manufacturing model. In this model, shortages are not permitted. The rate of consumption of the item is assumed to be uniform throughout the year. The item is produced and consumed simultaneously for a portion of the cycle time. During the remaining cycle time, only the consumption of the item takes place and the cost of production per unit is same irrespective of production lot size.

Let us suppose,

r = Annual demand in units

k = Production rate of the item (total number of units produced/year)

$C_o = Cost\ per\ set - up$

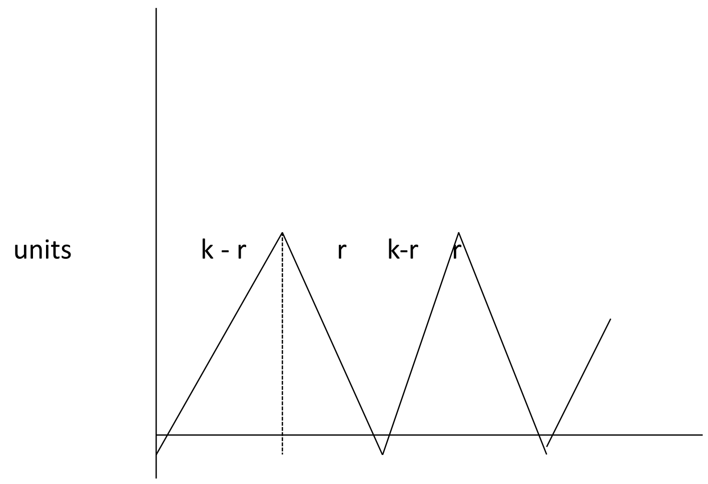
C_c =Carrying cost / unit /year

P = Cost of production/unit

t_1 = period of production as well as consumption of the item

t_2 = Cycle time (i.e. t = t_1 +t_2)

The operation of the manufacturing model without shortages is shown as in the following figure



Manufacturing model without shortages.

During the period t_1, the item is produced at the rate of k units per period and simultaneously it is consumed at the rate of r units per period. During this period, the inventory is built at the rate of k – r units per period. During the period t_2, the production of the item is discontinued but the consumption of the same item is continued. Hence, the inventory is decreased at the rate of r units per period during this time t_2. The various formula to be applied for this kind of situation are given below.

Economic batch quantity (EBQ or Q) = $\sqrt{\frac{2C_O r}{C_c\left[1-\left(\frac{r}{k}\right)\right]}}$

Period of production as well as consumption, $t_1 = \frac{Q}{k}$

Period of consumption only, $t_2 = \frac{Q\left[1-\left(\frac{r}{k}\right)\right]}{r} = \frac{(k-r)t_1}{r}$

Cycle time $t = t_1 + t_2$

Number of set-ups per year = $\frac{r}{Q}$

Example: An automobile factory manufactures a particular type of gear within the factory. This gear is used in the final assembly. The particulars of this gear are: demand rater 14,000 units/ year, production rate k = 35,000 units/year, set-up cost, C_o = Rs 500 per set-up and carrying cost,C_c= Rs. 15/unit/year. Find the economic batch quantity (EBQ) and cycle time.

Solution:

Applying the required formulae, we have the economic batch quantity

$$Q = \sqrt{\frac{2C_0 r}{C_c\left[1-\left(\frac{r}{k}\right)\right]}} = \sqrt{\frac{2\times500\times1400}{15\left[1-\left(\frac{14000}{35000}\right)\right]}}$$

= 1247.22 units

= 1248 (approx.)

Now, the period of production as well as consumption

$t_1 = \frac{Q}{k} = \frac{1248}{35000}$ = 0.0357 year = 0.4284 month

= 13 days (approx.)

and the period of consumption

$t_2 = \frac{Q}{r}(1 - \frac{r}{k}) = \frac{1248}{14000}(1 - \frac{14000}{35000})$ = 0.0535 year

= 0.642 month

= 20 days (approx.)

Therefore, the cycle time is

$t = t_1 + t_2$ = 13+20 = 33 days

Also

The number of set-ups per year = $\frac{r}{Q} = \frac{14000}{1248}$ = 11.22

Example: A company manufactures a low cost bearing which is used in its main product line. The demand of the bearing is 10,000 units per month and the production rate of the bearing is 25,000 units per month. The carrying cost of the bearing is Re. 0.02 per bearing per year and the set-up cost is Rs. 18 per set-up. Find the economic batch quantity (EBQ) and the cycle time (1)

Solution:

The given data are:

r = 10,000 bearings per month 1,20,000 bearings per year

k = 25,000 bearings per month 3,00,000 bearings per year

C_o = Rs. 18 per set-up

C_c = 0.02 per bearing per year

Therefore, the economic batch quantity (EBQ) is

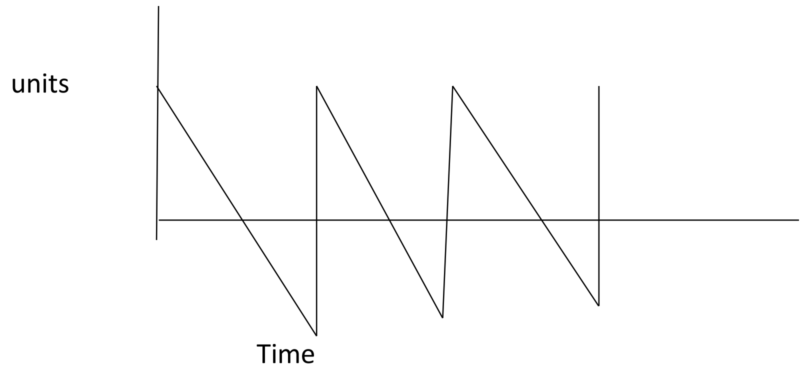
EBQ(Q) = $\sqrt{\frac{2C_0 r}{C_c\left[1-\left(\frac{r}{k}\right)\right]}} = \sqrt{\frac{2\times18\times120000}{0.02\times\left[1-\left(\frac{120000}{300000}\right)\right]}}$ = 18,973.65 units

= 18,974 units (approx.).

The cycle time, t= $\frac{EBQ}{r} = \frac{18,974}{120000}$ = 0.158 year = 1.9 months = 57 days.

Purchase Model with Instantaneous Replenishment and with Shortages

In this model, an item on order will be received instantaneously and it is consumed at a constant rate.The purchase price per unit is same irrespective of order size. If there is no stock at the time of receiving a request for the item, it is assumed that it will be satisfied at a later date with a penalty. This is called backordering. The model is shown as in the following figure.



Purchase model with shortages

The variables which are to be used in this model are

D = Demand/period

C_c = Carrying cost/unit/period

C_0= Ordering cost/order

C_s = Shortage cost/unit/period

Q = Order size

Q_1 = Maximum inventory

Q_2 = Maximum stock-out

t_1 = Period of positive stock

t_2 = Period of shortage

t = Cycle time (t_1 + t_2)

Optimal values of the above variables are:

$$Q = \sqrt{\frac{2C_oD}{C_c}\frac{C_s+C_c}{C_s}}$$

$$Q_1 = \sqrt{\frac{2C_oD}{C_c}\frac{C_s}{C_s+C_c}}$$

Q_2 = Q - Q1

$t = \frac{Q}{D}$

$t_1 = \frac{Q_1}{D}$

$t_2 = \frac{Q_2}{D}$

Where

Number of orders/period = $\frac{D}{Q}$

Example: The annual demand for a component is 7200 units. The carrying cost is Rs. 500/unit/ year, the ordering cost is Rs. 1500 per order and the shortage cost is Rs. 2000/unit/year. Find the optimal values of economic order quantity, maximum inventory, maximum shortage quantity, cycle time (t), inventory period (t_1) and, shortage period (t_2).

Solution:

We have

D = 7200 units/year

$C_c =$ Rs. 500/unit/year

C_0 = Rs 1500/order

C_s = Rs. 2000/unit/year

Therefore,

Economic order quantity Q = $\sqrt{\frac{2C_0D}{C_c}\frac{C_S+C_c}{C_S}} = \sqrt{\frac{2\times1500\times7200}{500}\frac{2000+500}{2000}}$

= 233 units (approx.)

Maximum inventory $Q_1 = \sqrt{\frac{2C_0D}{C_c}\frac{C_S}{C_{S+}C_c}} = \sqrt{\frac{2\times1500\times7200}{500}\frac{2000}{2000+500}}$

= 186 units (approx.)

Maximum stock-out Q_2 =Q - Q1 = 233 – 186 = 47 units

Cycle time t = $\frac{Q}{D} = \frac{233}{7200}$* 365 = 12 days (approx.)

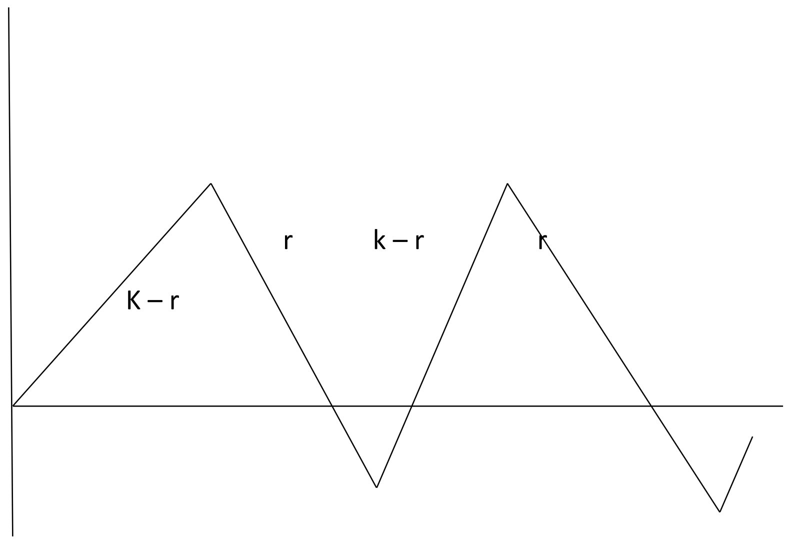
Period of positive stock = $t_1 = \frac{Q_1}{D} = \frac{186}{7200} \times 365$ = 10 days (approx.)

Period of shortage $t_2 = t - t_1$ = 12 – 10 = 2 days

Number of orders per year = $\frac{D}{Q} = \frac{7200}{233}$ = 30.9

Manufacturing Model with Shortages

In this model, an item is produced and consumed simultaneously for a portion of the cycle time. During the remaining cycle time, only the consumption of the item takes place. The cost of production per unit is the same irrespective of the production lot size. Stock-out is permitted in this model, and it is assumed that the stock-out units will be satisfied from the units which will be produced at a later date, with a penalty. This is called backordering. The operation of this model is shown in the following figure.



Manufacturing model with shortages

The variables which are used in this model are given below:

r = Demand of an item/period

k = Production rate of the item (number of units produced/period)

C_0 = Cost /set-up

C_c = cost/unit/period

C_s = Shortage cost/unit/period

t = Total cycle time

P = cost of production/unit

t_1= Period of production as well as consumption of the item satisfying period's requirement

t_2 = Period of consumption only

t_3 = Period of shortage

t_4 = Period of production as well as consumption of the item satisfying back order

The formulae for the optimal values of the above variables are presented below:

Economic batch quantity Q = $\sqrt{\frac{2C_0}{C_c}\frac{kr}{k-r}\frac{C_c+C_s}{C_s}}$

Maximum inventory Q_1 = $\sqrt{\frac{2C_0}{C_c}\frac{r(k-r)}{k}\frac{C_s}{C_c+C_s}}$

Maximum stock out Q_2 = $\sqrt{\frac{2C_0C_c}{C_{s(C_c+C_s)}}\frac{r(k-r)}{k}}$

$Q_1 = \frac{(k-r)}{k} Q - Q_2$

$t = \frac{Q}{r}$

$t_1 = \frac{Q\ 1}{k-r}$

$t_2 = \frac{Q_1}{r}$

$t_3 = \frac{Q_2}{r}$

$t_4 = \frac{Q_2}{k-r}$

Example: The demand for an item is 6000 units per year. Its production rate is 1000 units per month. The carrying cost is Rs. 50/unit/year and the set-up cost is Rs. 2000 per set-up. The shortage cost is Rs. 1000 per unit per year. Find various parameters of the inventory system.

Solution:

Here r = 6000 units/year

K = 1000×12 = 12,000 units/year

C_0 = Rs.2000/ set- up

C_c = Rs.50/unit/ year

C_s = Rs.1000/unit/year

Therefore,

$$Q\ (EBQ) = \sqrt{\frac{2C_0}{C_c}\frac{kr}{k-r}\frac{C_c+C_s}{C_s}} = \sqrt{\frac{2X\ 2000}{50}\frac{12000X6000}{12000-\ 6000}\frac{50+1000}{1000}} = 1004 \text{ units}$$

$$Q_2 = \sqrt{\frac{2C_0C_c}{C_{s(C_c+C_s)}}\frac{r(k-r\ )}{k}} = \sqrt{\frac{2\ X\ 2000X50}{1000(50+1000)}\frac{6000(12000-6000)}{12000}}$$

= 24 units

$Q_1 = \frac{(k-r)}{k}Q - Q_2 = \frac{(12000-6000)}{12000}X\ 1004 - 24 = 478$ units

$t = \frac{Q}{r}X\ 365 = \frac{Q}{r}X\ 365 = \frac{Q}{r}X\ 365 = 61$ days

$t_1 = \frac{Q\ 1}{k-r}X\ 365 = \frac{478}{12000-6000}X\ 365 = 29$ days

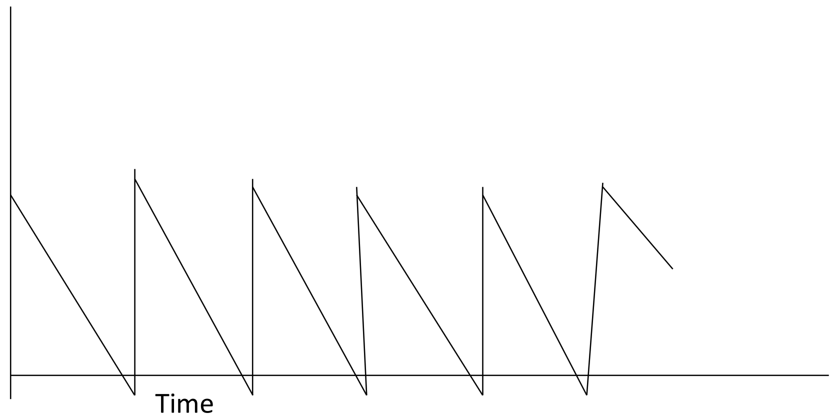
$t_2 = \frac{Q_1}{r}X\ 365 = \frac{478}{6000}X\ 365 = 29$ days

$t_3 = \frac{Q_2}{r}X\ 365 = \frac{24}{6000}X\ 365 = 1.5$ days

$t_4 = \frac{Q_2}{k-r}X\ 365 = \frac{24}{12000-6000}X\ 365 = 1.5$ days

Operation of inventory system:

Consider a purchase model of inventory system as shown in the following figure



D_{LT} = demand rate (perday) x lead time period (in days)

ROL = D_{LT}+ SS

SS = $K_{\propto}\sigma$

Example: In a firm, the distribution of demand of an item during a constant lead time follows normal distribution. The standard deviation of the demand of the item is 300 units. The firm wants to have a service level of 95 per cent. Find: How much safety stock should be carried out for the item? If the demand during lead time averages 2000 units, what is the appropriate reorder level?

Solution:

The safety stock is given by

SS = Kσ = (1.64) (300) = 492 units

Where K = 1.64 for 95% service level from standard normal table.

Reorder level (ROL) = DLT+SS=2000+492 = 2492 units

Example: The average demand of an item is 24000 units per year. The purchase price per unit is Rs. 1.25, the ordering cost is Rs. 25 per order and the carrying cost is 6% of the unit cost. The number of working days in a year is 320 days and the lead time is 10 days. The demand follows normal distribution and the standard deviation of the demand is 100 units. Find EOQ, safety stock and reorder level by assuming a confidence level of 95%.

Solution:

The data of the given problem are:

Average demand (D) = 24000 units per year

C_0 = Rs. 25 per order

P = Rs 1.25 per unit

C_C = 0.06 X 1.25 = Re. 0.075 per unit per year

Number of working days per year = 320 days

Lead time = 10 days

Standard deviation of demand (σ) = 100 units

Confidence level (1 – α) = 0.95 , α = 0.05

K_α= 1.64

Therefore, the economic order quantity

$$EOQ(Q) = \sqrt{\frac{2C_oD}{C_c}} = \sqrt{\frac{2\ X\ 25\ X\ 24000}{0.075}} = 4000 \text{ units}$$

Demand per day = (D/No. Of working days in one year) = $\frac{24000}{320} = 75\ units$ per day

Lead time demand, D_{LT} = Lead time X Daily demand = 10 X 75 = 750units

Safety Stock (SS) $= K_\propto \sigma$ = 1.64 X 100 = 164 units

Reorder level (ROL) =D_{LT} +SS=750+164=914 units.

QUANTITY DISCOUNT:

When an item is purchased in bulk, buyers are usually given discount in the purchase price of the item. Let i be the per cent of the purchase price accounted for carrying cost/unit/period. The discount may be a step function of purchase quantity as shown below.

Quantity	Purchase price per unit
$0 \leq Q1 < b1$	P_1
$b1 \leq Q2 < b2$	P2
$b2 \leq Q3 < b3$	P3
.	.
.	.
.	.
$bn - 1 \leq Qn$	

The procedure to compute the optimal order size for this situation is given in the following steps:

Step 1: Find the EOQ for the nth (last) price break.

$$Q\,n = \sqrt{\frac{2C_oD}{ip_n}}$$

If it is greater than or equal to b $_{n-1}$ then the optimal order size rho Q is equal to Q_n otherwise go to step 2.

Step 2: Find the EOQ for the (n – 1) price break.

$$Q_{n-1} = \sqrt{\frac{2C_oD}{ip_{n-1}}}$$

If it is greater than or equal to b $_{n-2}$ then compute the following and select the least cost purchase quantity as the optimal size; otherwise go to step 3.

(i) Total cost TC (Q_{n-1})

(ii) Total cost, TC (b_{n-1})

Step 3: Find the EOQ for the (n – 2)th price break.

$$Q_{n-2} = \sqrt{\frac{2C_oD}{ip_{n-2}}}$$

If it is greater than or equal to b n-3* then compute the following and select the least cost purchase quantity as the optimal order size; otherwise go to step 4:

(i) Total cost TC(Q_{n-2})

(ii) Total cost TC (b_{n-2})

(iii)Total cost TC (b_{n-1})

Step 4: Continue in this manner until $Q_{n-i} \geq b_{n-i-1}$. Then compare total costs

TC(Q_{n-i}), TC (b_{n-i}), TC(b_{n-i-1}),...,TC(b_{n-1}) Corresponding purchase quantities Q_{n-i} ,b_{n-1} ,b_{n-i+1} ,...,b_{n-1} , respectively. Finally, select the purchase quantity with respect to the minimum total cost as the optimal order size.

Example: Annual demand for an item is 6000 units. Ordering cost is Rs. 600 per order. Inventory carrying cost is 18% of the purchase price/unit/year. The price breakups are as shown below.

Quantity	Purchase price per unit
$0 \leq Q1 < 2000$	20
$2000 \leq Q2 < 4000$	15
$4000 \leq Q3$	9

Find the optimal order size

Solution:

Given that D = 6000/year,

C_0 = Rs . 600/order and

i = 18% of the purchase price unit/year.

Step 1: p3 = Rs. 9 Therefore,

Q 3 = $\sqrt{\frac{2C_oD}{ip_3}}$ = $\sqrt{\frac{2X\ 600X\ 6000}{0.18\ X\ 9}}$ = 2109 units

Since, Q3 < b2 (4000) and Q3 >b (2000), go to step 2.

Step 2: P2 = Rs.15. Therefore,

Q 2 = $\sqrt{\frac{2C_0D}{ip_2}}$ = $\sqrt{\frac{2\ X\ 600\ X\ 6000}{0.18\ X\ 15}}$ = 1633 units(approx.)

Since Q2 < b1 (2000), go to step 3.

Step 3: p1 = Rs 20. Therefore,

Q 1 = $\sqrt{\frac{2C_0D}{ip_1}}$ = $\sqrt{\frac{2\ X\ 600\ X\ 6000}{0.18\ X\ 20}}$ = 1415 units (approx.)

Since Q1 < b1 (2000), find the following costs and select the order size with respect to the least cost as the optimal order size.

TC(Q1) = (20)(6000) + $\frac{(600)(6000)}{1415} + \frac{(0.18)(20)(1415)}{2} = Rs. 125091$

TC(b1) = (15)(6000) + $\frac{(600)(6000)}{2000} + \frac{(0.18)(20)(2000)}{2}$ = Rs.94,500

TC(b2) = (9)(6000) + $\frac{(600)(6000)}{4000} + \frac{(0.18)(9)(4000)}{2}$ = Rs.58140

The least cost is Rs. 58,140. Hence, the optimal order size is b2 which is equal to 4000 units.

PROJECT SCHEDULING WITH CPM & PERT

Introduction

A project is any task which has definable beginning and definable end, and requires investment and expenditure of one or more resources in each of the separate but interrelated and inter-dependent activities which must be accomplished to achieve the objectives for which the task or project was instituted.

It is very much essential to effectively manage the projects through proper planning, scheduling and control as project requires a heavy investment, and is associated with risk and uncertainties.

Network scheduling is a technique used for planning and scheduling large projects in the fields of construction, maintenance, fabrication and any other areas. This technique is the method of minimising the bottlenecks, delays and interruptions by determining the critical factors and coordinating various activities.

There are two basic planning and control techniques. They are critical path method (CPM) and programme Evaluation and Review techniques (PERT).

Objectives Of Network Analysis

1. A powerful co-ordinating tool for planning, scheduling and controlling of projects.

2. Minimisation of total project cost and time

3. Effective utilisation of resources and minimisation of effective resources.

4. Minimisation of delays and interruption during implementation of the project.

Applications Of Network Analysis (PERT & CPM)

1. Research and development projects

2. Equipment maintenance and overhauling

3. Construction projects (building, bridges, dams)

4. Setting up new industries

5. Planning and lunching of new products

6. Design of plants, machines and systems

7. Shifting the manufacturing location from one location to another

8. Control of production in large job shops

9. Market penetration programmes

10. Organisation of big programmes, conferences etc.

Basic Concepts In Network

(a) Network: It is a graphical representation of the project and it consists of series of activities arranged in a logical sequence and show the interrelationship between the activities

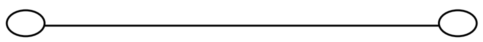
(b) Activities: An activity is a physically identifiable part of the project, which consumes time and resources. Each activity has a definite start and end. Activity is represented by an arrow (è).

(c) Event: An event represents the start or the completion of an activity. The beginning and end points of an activity are events.

Example: Machining a component is an activity

Start Machining is an event

Machining completed is an event



Tail Event Head Event

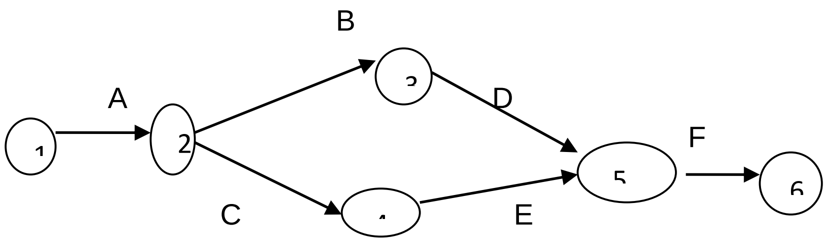
In a network a number of activities may terminate in to single node, called merge node and a number of activities may emanate from a single node called burst node.

The merge event, and burst events are shown in the following figure

(a) Merge Event (b) Burst Event

Merge and burst events.

(d) Predecessor and Successor Activities: All those activities, which must be completed before starting the activity under consideration are called its predecessor activities. All the activities which have to follow the activity under consideration are called its successor activities.



Network Diagram

In the above figure, Activities 2-3 and 2-4 are immediate successors to activity 1-2
Activities 2-3 and 2-4,3-5, 4-5 and 5-6 are its successor activities.
Activities 1-2, 2-3, are predecessors to activity 3-5 and 2-3 is the immediate predecessor. The precedence relationship between the activities can be expressed as –

Activity	Immediate predecessor
A	-
B	A
C	A
D	B
E	C
F	D,E

(e) Path-An unbroken chain of activities between two events is called a path.
A-B-D-F is a path connecting events (1) and (6). There are number of paths traced through the network.

(f) Dummy Activity - An activity which depicts the dependency or relationship over the other but does not consume time or resources. It is used to maintain the logical sequence. It is indicated by a dotted line.
A dummy activity shown in the following figure

Consider the network of A activities P,Q, R & S. activity C is preceded by activities A and B. While activity D is preceded by B only.

(a) Wrong (b) correct

Numbering of Events (FULKERSONS RULE)

The Steps involved in numbering the events as per the Fulkerson's rule are;

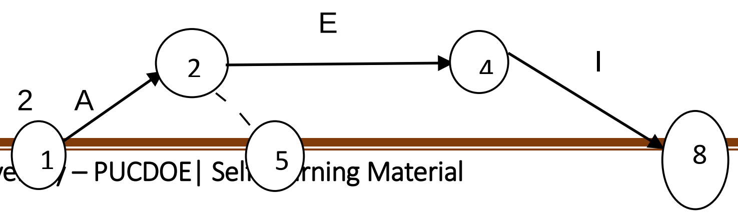
1. The initial event which has all outgoing arrows with no incoming arrow is numbered "1"

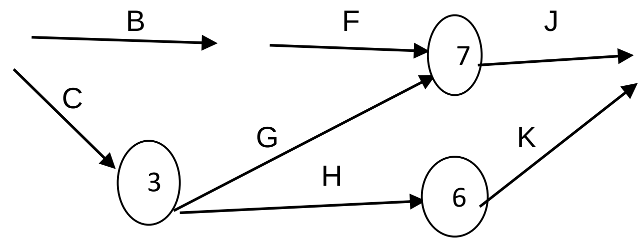
2. Delete all the arrows coming out from node "1". This will convert some more nodes (at least
one) into initial events Number these events 2,3-etc.

3. Delete all the arrows going out from these numbered events to create more initial events
Assign next numbers to these events.

4. Continue until the final or terminal node which has all arrows coming in, with no arrow going
out is numbered

The following figure shows numbering of the network





Numbering the network

Critical Path Method

In critical path method (CPM) the activity times are known with certainty. For each activity earliest start time (EST) and latest start times (LST) are computed.

"The path with the longest time sequence is called critical path. The length of the critical path determines the minimum time in which the entire project can be completed. The activities on the critical path are called "Critical activities ".

The time analysis in network is done with an objective of;

- Determining the completion time for the project
- Earliest time when each activity can start
- Latest time when each activity can start without delaying the total project
- Determining float for each activity
- Identification of the critical activities and critical path.

FORWORD PASS COMPUTATIONS (Earliest event time)

Forward pass computation gives the earliest expected start and finish times for each activity (indirectly earliest occurrence time for each event). The computations start from the initial node and move to the end node. Forward pass computation, starts with an assumed earliest occurrence of zero for the initial event.

(a) Earliest start time (ES): It is the earliest event time of the tail end event-

$$\bar{E}S_{ij} = E_i$$

Where, $\bar{E}S_{ij}$ = Earliest start for an activity (i,j)

Ei = Earliest event occurrence time of event i.

(b) **Earliest Finish time** of an activity (EF) is the earliest Starting time + Activity time

EF_{ij}_= $ES_{ij} + t_{ij}$ = Time Estimate of activity

(c) Earliest event time for event 'j' is the maximum of the earliest finish times of all activities ending in to that event.

E_j = Maximum of ($ES_{ij} + t_{ij}$)= Maximum ($E_I + t_{ij}$)

BACKWORD PASS COMPUTATIONS: (Latest allowable time)

The latest event times specifies the time by which all activities entering in to that event must be completed without delaying the total project.

(a) Latest finish time for an activity (i, j) equals the latest event time of event 'j - $LF_{ij} = L_{ij}$

(b) Latest starting time of activity (i, j) the latest completion time of (i, j) minus the activity time

ie. $LS_{ij} = LF_{ij} - t_{ij}$

(c) Latest event time for event 'i' is the mininum of the latest start time of all activities originating from that event.

$L_i = \text{Maximum}_j\ (LS_{ij})$

$= \text{Minimum}\ (LF_{ij} - t_{ij}) = \text{Min}_j (L_j - t_{ij})$

(d) Slack-The slack of an event is the difference between the latest and earliest event times Slack (i) = Li - Ei

The Events with Zero slack time are known as critical events.

(e) Floats

(i) Total float - It is concerned with overall project duration. It is defined as "the amount of time by which completion of an activity can be delayed beyond earliest expected completion time without affecting overall project duration time."

Total float for an activity (i, j) is the difference between the latest start time and earliest start time for that activity.

$TF_{ij} = LS_{ij} - ES_{ij}$

$= (L_j - E_i) - t_{ij}$

Where Ei =earliest expected completion time of tail event

= Earliest starting time for an activity (i,j)

L= Latest allowable completion time of head event

= latest finish time for activity (i,j)

(ii) Free Float - It is the time by which the completion of an activity can be delayed beyond the

earliest finish time without affecting the earliest start of a subsequent activity.

Free float for an activity

$=(E_j - E_i) - E_{ij}$

(iii) Independent Float - It is the amount of time by which the start of an activity can be delayed without affecting earliest start time of any immediately following activities assuming that the preceding activity has finished at its latest finish time. Independent float is given by;

$$(Ei-Li)-Eij$$

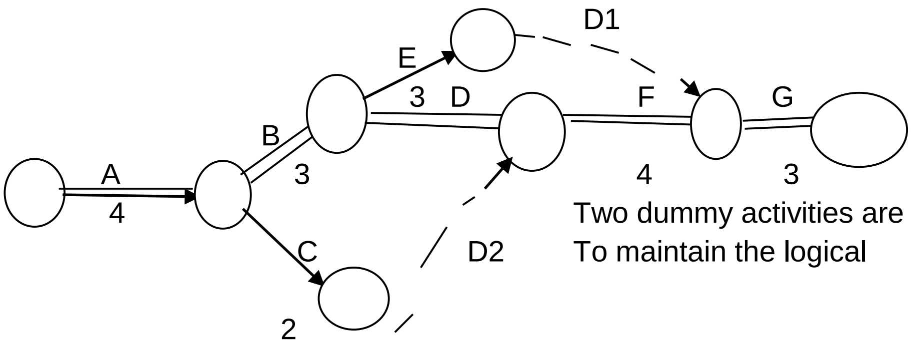
Problem 1

The activity details and their predecessors are given below along with their activity times Construct the network diagram

Activity	predecessors	Activity time(weeks)
A	-	4
B	A	3
C	A	2
D	B	5
E	B	3
F	C,D	4
G	E,F	3

Solution.

The network diagram is shown in the flowing figure



Two dummy activities are used To maintain the logical sequence

Network Diagram

The slack for the activities is calculated as shown in table below

Activity	Activity time	ES	EF	LS	LFT	Slack
A	4	0	4	0	4	0
B	3	4	7	4	7	0
C	2	4	6	8	12	2
D	5	7	12	7	12	0
E	3	7	10	9	12	3
F	4	12	16	12	16	0
G	3	16	19	16	19	0

Table: Computation of Slack

The critical path is one that connects the activities with zero slack. The critical paths

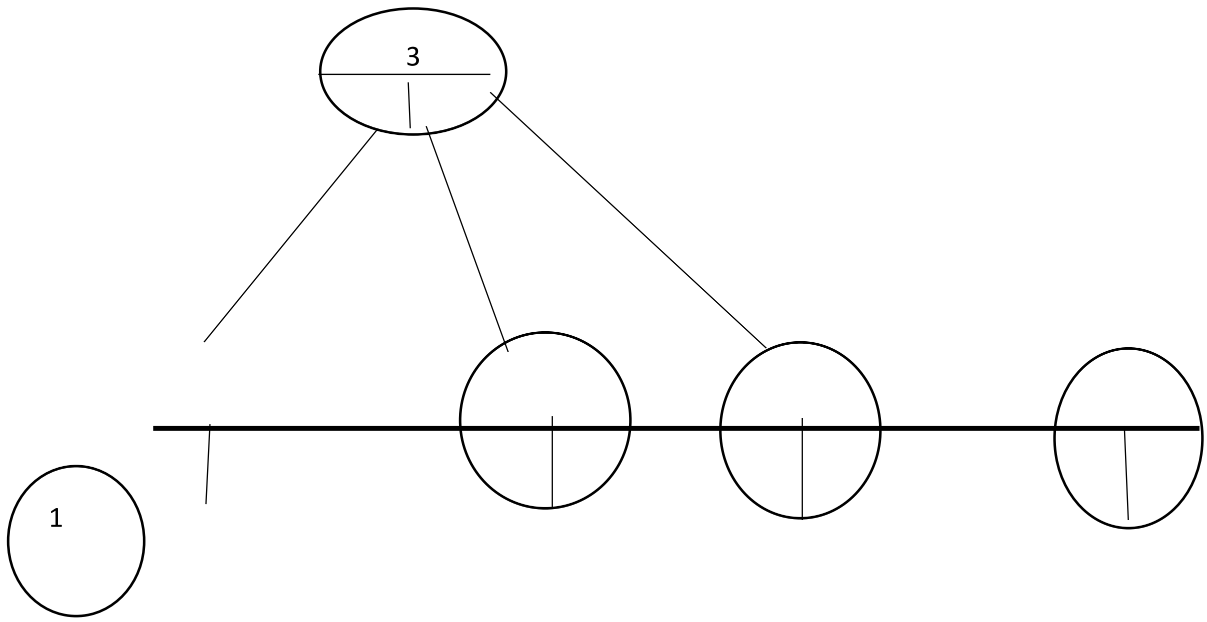
A-B-D-F-G

Problem 2.
The activities involved in a small project are given below along with relevant information.

Construct the network and find the critical path
Find the floats for each activity.

Activity	1-2	1-3	2-3	2-4	3-4	4-5
Duration	20	25	10	12	6	10

Solution.
The network diagram along with timing is shown in the following figure



Critical path is 1-2-3-4-5

The Floats are calculated as shown in table below

Activity	Duration	Earliest		Latest		Float		
		Start	Finish	Start	Finish	Total	Free	Independent
1-2	20	0	20	0	20	0	0	0
1-3	25	0	25	5	30	5	5	5
2-3	10	20	30	20	30	0	0	0
2-4	12	20	32	24	36	4	4	4
3-4	6	30	36	30	36	0	0	0
4-5	10	36	46	36	46	0	0	0

Programme Evaluation And Review Techniques (PERT)

Critical path method has only one time estimate. So it does not consider uncertainty in time. In reality the duration of activities may not be deterministic (certain) in all cause.

PERT takes in to account the uncertainty of activity times. It is a probabilistic model with
Uncertainty in activity duration

PERT makes use of three estimates of time;

(1) Optimistic time (To)

(ii) Most likely time (Tm)

(iii) Pessimistic time (Tp)

Optimistic time (To) is the shortest possible time, if everything goes perfectly without any complications. It is an estimate of minimum possible time to complete the activity under ideal condition.

Pessimistic time (Tp) is the longest time taking in to consideration all odds. This is the time estimate if everything goes wrong.

Most likely time (Tm) is the best estimate of the activity time. This lies between the optimistic and pessimistic time estimates.

The three time estimates to, tp and tm are combined to develop expected time (Te) for an activity. The fundamental assumption in PERT is that the three estimates of time follow a ß (Beta) distribution.
Probability of Completion of the Project within a Scheduled Time

The probability of completing a project is given by the probability of occurrence of the end event of the network. The probability distribution is assumed to be the normal distribution.

The expected time (Te) is given by

$$Te = \frac{T0+4Tm+Tp}{6}$$

The standard deviation of the time required to complete each activity

$$\text{Stan.Deviation} \quad (\sigma) = \frac{Tp-To}{6}$$

Standard deviation of the time up complete the project

$$= \frac{Tp1-To1}{6} + \frac{Tp2-To2}{6} \pm ------ + \frac{Tpn}{6}$$

The probability of completion of the project within scheduled time

The probability completing a project is given by the probability of occurrence of the end
Event of the network. The probability distribution is assumed to be the normal distribution.

The probability of completion of the project within scheduled is computed as;

1. Calculate the mean of the event time (te) by adding the times of the activities along the critical path leading to the event.

2. Calculate the variance of the event time by adding up the variances of the activities on the critical path. Take the squire root of this variance to get T (standard deviation)

3. Compute standard normal variate, Z= Ts-Te /σ т
From tables of normal curve, the value corresponding to Z gives the required probability,

Problem:

A small project is composed of time activities whose time estimate are givenbelow:

Activity	A	B	C	D	E	F	G	H	I
Optimistic time	2	2	4	2	2	3	2	5	3
Most likely time	2	5	4	2	5	6	5	8	6
Pessimistic time	8	8	10	2	14	15	8	11	15

Activities A, B and C can start simultaneously. Activity D follows activity A while E follows & Activity D and E are followed by activity G while F is dependent on C, H depends on D and E, while I depends on F and G.

(1) Construct the network.

(ii) Find the expected duration and variance of each activity.

(iii) Calculate the slack for each event.

(iv) What is the critical path and expected project duration of the project.

(v) If the project due date is 28 days, what is the probability of not meeting the due date.

(vi) What should be the project duration for the probability of completion of 95%?

Solution:

Expected time Te = $\frac{a+b+4m}{6}$

Variance V = $(\frac{b-a}{6})^2$

Activity	Expected time Te	Variance V
A	3	1
B	5	1
C	5	1
D	2	0
E	6	4
F	7	4
G	5	1
H	8	1
I	7	2

Critical path is 1- 3- 5-9 – 10

B- E – G – I

Project duration 23 days

Standard deviation $\sigma = \sqrt{V_{ij}} = 2.83$

(i) Z= $\frac{Ts-Te}{6} = \frac{28-23}{6} = 0.833$

Probability of completion 78.8%

(ii) Z=1.65 = $\frac{Ts-Te}{6}$

Schedule time = 33 days.

Time Cost Trade Off (Crashing)

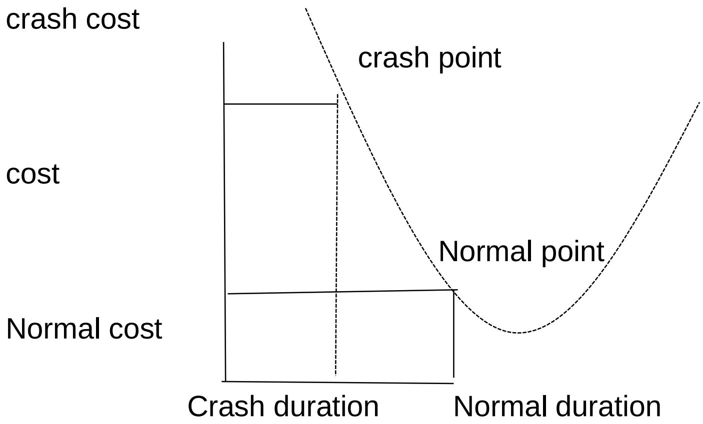
The total overall cost will consist of direct and indirect costs. The indirect cost consists of overheads, office expenditures, administrative expenses, and cost penalty for delay in work etc.

The indirect time varies with time. Any reduction in project time means reduction in indirect costs.

Direct costs include cost of materials, machinery used in the project and payments towards labour and subcontracting.

It is assumed that for each activity, these are an activity duration for which the direct cost is nimum. If activity results in more than this time, more resources and hence more funds are required For instance a point will be reached beyond which no further reduction in time will be possible irrespective of resources spent.

The time for the activity at minimum cost is called normal time and the minimum time for the activity is called crash time. The costs associated with these times are called Normal and crash costs. A linear relationship is assumed between time and cost as shown in the following figure.



Time cost relationship

Cost slope(CS) = $\frac{Crash\ cost - Normal\ cost}{Normal\ time - Crash\ time}$

Procedure for crashing

The process of reducing the project duration is called crashing. The crashing is at the cost of extra resources

i.e. at an extra cost.

The procedure for project crashing is;

1. Construct the network diagram and for a given network diagram find the critical path

2. Calculate the cost slope for different activities.

2. Crashing the network - crashing the activities in critical paths as per the ranking activity having lower cost slope would be crashed first to the maximum possible extent.

4. Determine the total cost of the project

Problem

. From the activity details given below, determine the optimal project duration

Table: Cost Time and Cost Slopes.

Normal		**crash**			**Cost slope**
Activity	**time**	**Cost**	**Time**	**Cost**	
1-2	8	100	6	200	50
1-3	4	150	2	350	100
2-4	2	50	1	90	40
2-5	10	100	5	400	60
3-4	5	100	1	200	25
4-5	3	80	1	100	10

Indirect Cost RS. 70 per day

Solution.

The network is shown in the following figure along with earliest and latest starts.
Critical path = 1-2-5
= 8 + 10 = 18days .
Direct cost of the project = Rs. 580
Indirect cost of the project = 18 x 70 = Rs.1260
Total project cost = Rs. 1840.

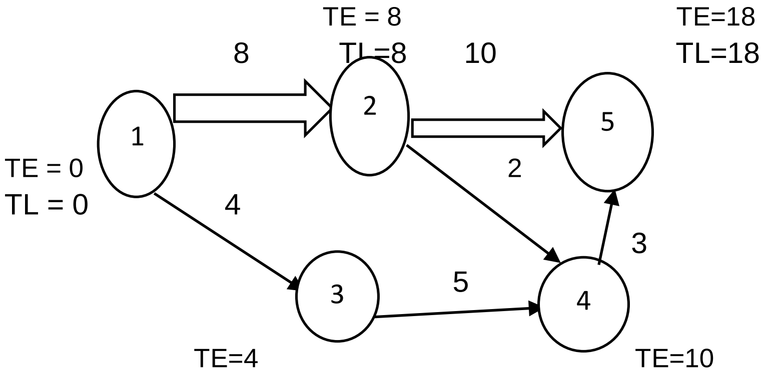
The cost slope $= \frac{Crash\ cost - Normal\ Cost}{Normal\ time - Crash\ time}$

Instead of crashing all the activities, only those activities whose cost slope is less than the indirect cost.

Along the critical path activities | - 2 - 5 , both activities 1-2&2-5 have cost-slopes less than indirect cost.

The minimum cost slope is associated with 1 - 2 and it can be crashed by two days.

Project duration is reduced to 16 days while critical path remains unchanged
Direct Cost = 580+2x50=680
Indirect Cost 70×16=1120
Total Cost 680+1120 = 1800



TL=10 TL=15

Network Diagram with ES and LS

Next Crash the activity 2-5 by 5 days

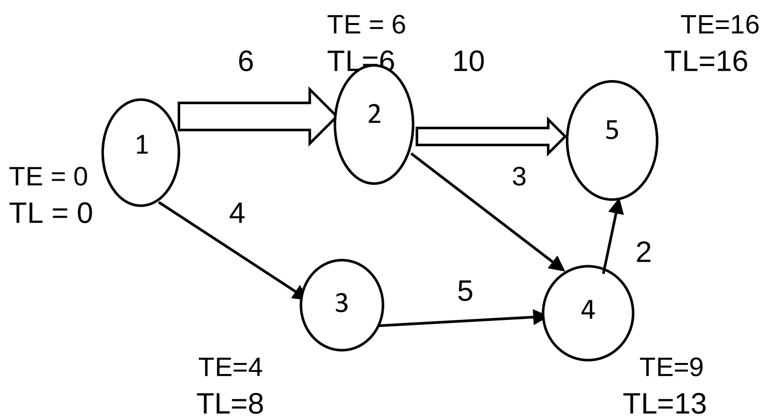
The critical path is changed to 1-3-4-5. This is the shown in the following figure.
Direct Cost = 680 + 5 * 60 = 980
Indirect Cost = 12 * 70 = 840
Total Cost= 980-840=1820

Activity on new critical path (Slope less than 70 are 3-4&4-5) crash on new critical the activity
4-5 by days

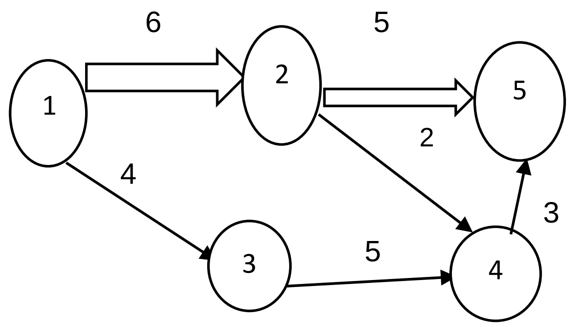
Critical path again chances to 1-2-5
Total Cost = 980+2x10 +11×70=1770

This is shown in the following figure.

The project cannot be crashed any further as the activities on the critical path are already crashed. From the network diagram, the activity 4-5 can be expended by one day without affecting the project duration.



Network after crashing



This will save crashing cost of activity 4-5 for one day
Total Cost=1760

Optimal Project duration = 11 days
Optimal Project Cost = 1760.

Comparison Between CPM And PERT –

CPM	PERT
1 CPM is activity oriented.	PERT is event oriented
2 CPM is used when the activity times are deterministic	PERT uses a probabilistic times
3. One time estimate	Three time estimates (a) Optimistic, (b) Most (c)
4. CPM directly introduces cost concept analysis	PERT indirectly for costs
5. CPM is a planning device	PERT is a control device.

QUEUEING THEORY

INTRODUCTION

In many real-world applications such as railways and airlines reservation counters, bank counters, gasoline stations, etc., incoming customers become part of the respective queueing system. In fact, waiting for service has become an integral part of our daily life, albeit at a considerable cost most of the times. However, the adverse impact of the queueing up phenomena can be brought down to a minimum by applying various queueing models.

In general, the queueing system consists of one or more queues and one or more servers, and operates under a set of procedures. Let us consider the reservation counter of an airlines where customers from different parts of the world/country arrive and wait at the reservation counter. Depending on the server status, the incoming customer either waits at the queue or gets the turn to be served. If the server in the reservation counter is free at the time of arrival of a customer, the customer can directly enter into the counter for getting service and then leave the system. In this process, over a period of time, the system may experience 'customer waiting' and/or 'server idle time'. In any service system/manufacturing system involving queueing situation, the objective is to design the system in such a manner that the average waiting time of the customers is minimized and the Percentage utilization of the server is maintained above a desired level.

The various types of queueing system in many service/manufacturing situations are described in the following table

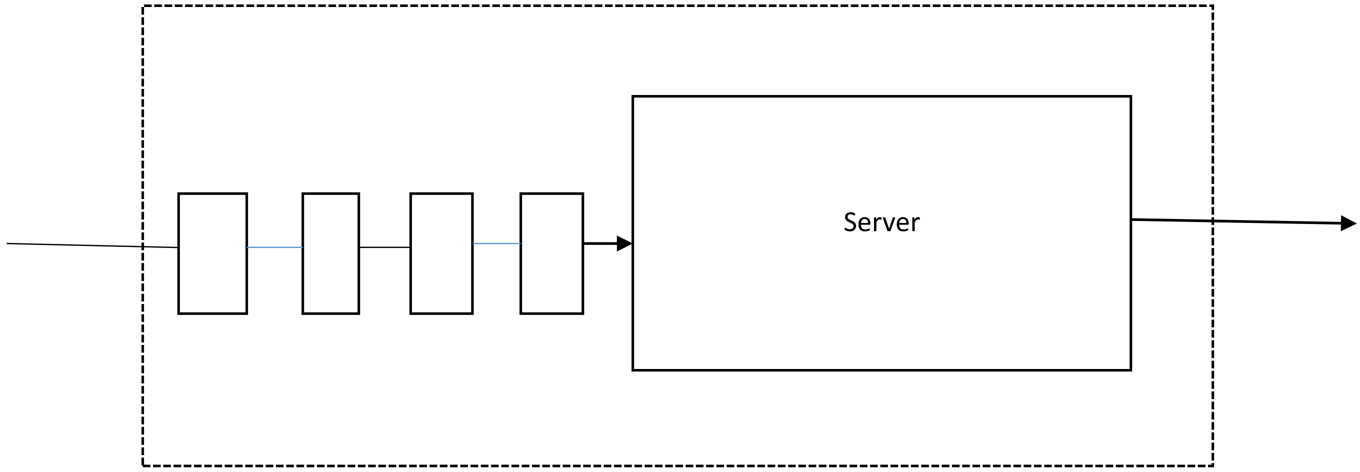
Application Areas of Queueing System

Example	Members of queue	Server(s)
Bank counter	Account holders	Counter clerk
Toll gate	Vehicles	Toll collectors
Ration shop	Ration card holders	Shop clerk
Main frame computer centre	Programs	Computer
Library	Students	Counter clerk
Traffic signal	Vehicles	Signal point
Final inspection station of T.V.	Assembled T.V. sets	Inspector
Airport runways	Planes	Runways
Telephone booth	Customers	Telephones

In addition to these examples, many subsystems of production, finance, personnel and marketing functions of an organization can be modelled as queueing systems for management decision making

TERMINOLOGIES OF QUEUEING SYSTEM

A schematic representation of a simple queueing system that consists of a queue and a service station is shown in the following figure



Queueing system

Customers who come to the system to get the required service will directly enter the service station without waiting in the queue if the server is free at that point of time. Otherwise, they will wait in the queue till the server becomes free. But in reality, there may be variations of the system as given below.

1. The number of queues may be more than one. If there is a queue for male as well as for female customers, then generally, alternate mode of selecting customers from each queue is followed.
2. The number of servers may be more than one. This is an example of parallel counters for providing service.
3. Sometimes, the service may be provided in multistages in sequential order. This type of system is known as queues in tandem.

Bulk arrival: Generally, it is assumed that the customers arrive into the system one by one. But, in some reality, customers may arrive in groups. Such arrival is called as bulk arrival.

Jockeying: If there is more than one queue, the customers from one queue will be tempted to join another queue because of its smaller size. This behaviour of the customers is known as queue jockeying.

Balking: If the queue length appears very large to a customer he/she may not join the queue. This property is known as balking of customers.

Reneging: Sometimes, a customer who is already in the queue will leave the queue in anticipation of longer waiting time. This kind of departure from the queue without receiving the service is known as reneging.

The list of variables which is used in queueing models is presented below:

n = Number of customers in the system

C = Number of servers in the system

$P_n(t)$ = Probability of having n customers in the system at time t.

P_n = Steady-state probability of having n customers in the system

P_o = Probability of having 0 customer in the system

L_q = Average number of customers waiting in the queue

L_s = Average number of customers waiting in the system (in the queue and in the service station)

W_q = Average waiting time of customers in the queue

W_s = Average waiting time of customers in the system (in the queue and in the service station)

δ = Arrival rate of customers

μ = Service rate of the server

φ = Utilization factor of the server

δ_{eff} = Effective arrival rate of customers

M = Poisson distribution

N = Maximum number of customers permitted in the system. Also, it denotes the size of the calling source of the customers

GD = General discipline for service. This may be first-in-first-serve (FIFS), last-in-first-serve (LIFS), random order (RO), etc.

EMPIRICAL QUEUEING MODELS:

The basic queueing models can be classified into six categories using Kendall notation which in turn uses six parameters to define a model such as (P/Q/R):(X/Y/Z). The parameters of this notation are:

P = Arrival rate distribution

Q = Service rate distribution

R = Number of servers

X = Service discipline

Y = Maximum number of customers permitted in the sysyem

Z = Size of the calling source of the customers

The basic six queueing models as per this classification are as shown in the following.

- (M/M/1):(GD/∞/∞)
- (M/M/C):(GD/∞/∞)
- (M/M/1):(GD/N/∞)
- (M/M/C):(GD/N/∞)
- (M/M/1):(GD/N/N)
- (M/M/C):(GD/N/N)

(M/M/1):(GD/∞/∞) Model:

The parameters of this model are given as follows:

1. Arrival rate follows Poisson distribution.
2. Service rate follows Poisson distribution.
3. Number of servers is one.
4. Service discipline is general discipline.
5. Maximum number of customers permitted in the system is infinite.
6. Size of the calling source is infinite.

The steady-state formula to obtain the probability of having n customers in the system P_n, and the formulas for P_0, L_s, L_q, W_s and W_q are presented below:

$P_n = (1 - \;)\ \varphi^n$, n = 0, 1, 2, 3,…. ∞ where $\varphi = \frac{\delta}{\mu} < 1$

$L_s = \frac{\varphi}{1-\varphi}$

$L_q = L_s - \frac{\delta}{\mu} = \frac{\varphi^2}{1-\varphi}$

$W_s = \frac{L_s}{\delta} = \frac{1}{(1-\varphi)\mu} = \frac{1}{\mu-\delta}$

$W_q = \frac{L_q}{\delta} = \frac{\varphi}{\mu-\delta}$

Example : The arrival rate of customers at a banking counter follows Poisson distribution with a mean of 45 per hour. The service rate of the counter clerk also follows Poisson distribution with a mean of 60 per hour.

(a) What is the probability of having 0 customer in the system (P_0)?
(b) What is the probability of having 5 customers in the system (P_5})?
(c) What is the probability of having 10 customers in the system (P_{10})
(d) Find L_s, L_q, W_s and W_q.

Solution :

We have the following data:

Arrival rate, δ= 45 per hour

Service rate, μ = 60 per hour

Utilization factor, $\varphi = \frac{\delta}{\mu} = \frac{45}{60} = 0.75$

(a) $V_0 = 1 - \varphi = 1 - 0.75 = 0.25$

(b) $V_S = (1 - \varphi)\,\varphi^5 = (1 - 0.75)\,0.75^5 = 0.0593$

(c) $P_{10} = (1 - \varphi)\,\varphi^{10} = (1 - 0.75)\,0.75^{10} = 0.0141$

(d) $L_s = \frac{\varphi}{1-\varphi} = \frac{0.75}{(1 - 0.75)} =$ 3 customers

$L_q = \frac{\varphi^2}{1-\varphi} = \frac{0.75^2}{(1 - 0.75)} =$ 2.25 customers

$W_s = \frac{1}{\mu-\delta} = \frac{1}{60-45} = 0.067$ hour

$W_q = \frac{\varphi}{\mu-\delta} = \frac{0.75}{60-45} = 0.05$ hour.

(M/M/C):(GD/∞/∞) Model

The parameters of this model are as follows:

(i) Arrival rate follows Poisson distribution.

(ii)Service rate follows Poisson distribution.

(iii) Number of servers is C.

(iv)Service discipline is general discipline.

(v) Maximum number of customers permitted in the system is infinite.

(vi) Size of the calling source is infinite.

The steady-state formula to obtain the probability of having n customers in the system P_n and the formula for P_0 ,L_s, L_q, W_s and W_q are presented below.

$P_n = \frac{\varphi^n}{n!} P_0$, $0 \le n \le C$

$= \frac{\varphi^n}{C^{n-c} C!} P_0, \quad n > C$

$P_0 = \left\{ \sum_{n=0}^{C-1} \frac{\varphi^n}{n!} + \frac{\varphi^C}{C!\left[1-\left(\frac{\varphi}{C}\right)\right]} \right\}^{-1}$

$L_q = \frac{\varphi^{C+1}}{(C-1)!(C-\varphi)^2} P_0 = \frac{C\varphi P_C}{(C-\varphi)^2}$

$L_s = L_q + \varphi$

$W_q = \frac{L_q}{\delta}$

$W_s = W_q + \frac{1}{\mu}$

Now the formula for P_0 and L_q under special conditions are :

$P_0 = 1 - \varphi$, $L_q = \frac{\varphi^{C+1}}{C^2}$, where $\varphi \ll 1$ and

$P_0 = \frac{(C-\varphi)(C-1)!}{C^C}$ and $L_q = \frac{\varphi}{C-\varphi}$, where $\frac{\varphi}{C} \approx 1$

Example

At a central warehouse, vehicles arrive at the rate of 18 per hour and the arrival rate follows Poisson distribution. The unloading time of the vehicles follows exponential distribution and the unloading rate is 6 vehicles per hour. There are 4 unloading crews. Find the following:

(a) P_0 and p_3

(b) L_q, L_s, W_q and W_s

Solution:

We have Arrival rate, δ = 18 per hour

Unloading rate, μ = 6 per hour

Number of unloading crews, c = 4

$$\varphi = \frac{\delta}{\mu} = \frac{18}{6} = 3$$

(a) $P_0 = \left\{\sum_{n=0}^{C-1} \frac{\varphi^n}{n!} + \frac{\varphi^C}{C!\left[1-\left(\frac{\varphi}{C}\right)\right]}\right\}^{-1}$

$= \left\{\sum_{n=0}^{3} \frac{3^n}{n!} + \frac{3^4}{4!\left[1-\left(\frac{3}{4}\right)\right]}\right\}^{-1}$

= 0.0377

$P_n = \frac{\varphi^n}{n!} P_0$, $\quad 0 \le n \le C$

$P_3 = \frac{\varphi^3}{3!} P_0$, $\quad 0 \le n \le C$

$= \frac{3^3}{3!} 0.0377 = 0.1697$

(b) $L_q = \frac{\varphi^{C+1}}{(C-1)!(C-\varphi)^2} P_0 = L_q = \frac{3^5}{3!\ X\ 1} 0.0377 = 1.53 \approx 2$ vehicles

$L_s = L_q + \varphi$ = 1.53 + 3 = 4.53 ≈ 5 vehicles

$W_q = \frac{\text{Lq}}{\delta} = \frac{1.53}{18}$ = 0.085 hour = 5.1 minutes

$W_s = W_q + \frac{1}{\mu}$ = 0.085 + $\frac{1}{6}$ = 0.252 hour = 15.12 minutes

(M/M/1):(GD/N/∞) Model:

The parameters of this model are as follows:

(i) Arrival rate follows Poisson distribution.

(ii)Service rate follows Poisson distribution.

(iii) Number of servers is C.

(iv)Service discipline is general discipline.

(v) Maximum number of customers permitted in the system is infinite.

(vi) Size of the calling source is infinite.

The steady-state formula to obtain the probability of having n customers in the system P_n and the formula for P_0, L_s, L_q, W_s and W_q are presented below.

$$P_n = \frac{1-\varphi}{1-\varphi^{n+1}}\varphi^n , \quad \varphi \neq 1 \text{ and } n = 0,1,2,3.....N$$

$$= \frac{1}{N+1} \quad \varphi = 1$$

$$L_s = \frac{\varphi[1-(N+1)\varphi^N+N\varphi^{N+1}]}{(1-\varphi)(1-\varphi^{N+1})}, \quad \varphi \neq 1$$

$$= \frac{N}{2}, \varphi = 1$$

$$\delta_{eff} = \delta(1-P_N) = \mu(L_s - L_q)$$

$$L_q = L_s - \frac{\delta_{eff}}{\mu} = L_s - \frac{\delta(1-P_N)}{\mu} ,$$

$$W_q = \frac{Lq}{\delta_{eff}} = \frac{Lq}{\delta(1-P_N)}$$

$$W_s = W_q + \frac{1}{\mu} = \frac{Ls}{\delta_{eff}} = \frac{Ls}{\delta(1-P_N)}$$

Example:

Cars arrive at a drive-in restaurant with a mean arrival rate of 24 cars per hour and the service rate of the cars is 20 cars per hour. The arrival rate and the service rate follow Poisson distribution. The number of parking space for cars is only 4. Find the standard results of this system.

Solution:

Here

Arrival rate, δ= 24 cars per hour

Service rate, μ = 20 cars per hour

N = 4

$\varphi = \frac{\delta}{\mu} = \frac{24}{20} = 1.2$

Therefore, we get

$$L_s = \frac{\varphi[1-(N+1)\varphi^N+N\varphi^{N+1}]}{(1-\varphi)(1-\varphi^{N+1})} = \frac{1.2[1-(4+1)1.2^4+4\times1.2^5]}{(1-1.2)(1-1.2^5)} = 2.36 \text{ cars}$$

$$P_N = \frac{1-\varphi}{1-\varphi^{N+1}}\varphi^N = \frac{1-1.2}{1-1.2^5}\times 1.2^4 = 0.2787$$

The other results are:

$\delta_{eff} = \delta(1 - P_N) = 24(1-0.2787) = 17.3112$ per hour

$L_q = L_s - \frac{\delta_{eff}}{\mu} = 2.36 - \frac{17.3112}{20} = 1.494$ cars

$W_q = \frac{Lq}{\delta_{eff}} = \frac{1.494}{17.3112} = 0.0863 hour = 5.2\ minutes$

$W_s = \frac{Ls}{\delta_{eff}} = \frac{2.36}{17.3112} = 0.1363\ hour = 8.2\ minutes$

(M/M/C)(GD/N/∞) Model (for C ≤ N)

The parameters of this model are defined as follows:

(i) Arrival rate follows Poisson distribution.
(ii) Service rate follows Poisson distribution.
(iii) Number of servers is C.
(iv) Service discipline is general discipline.
(v) Maximum number of customers permitted in the system is N.
(vi) Size of the calling source is infinite.

In this model, the following assumptions are made:

$\delta_n = \delta$, $0 \le n \le N$

$= 0$, $n \ge N$

and

$\mu_n = n\mu$, $0 \le n \le C$

$= C\mu, \ C \le n \le N$

The steady-state formula to obtain the probability of having n customers in the system P_n , and the formulas for P_0 ,L_s, L_q, W_s and W_q are presented below:

$$P_n = \frac{\varphi^n}{n!} P_0, \qquad 0 \le n \le C$$

$$= \frac{\varphi^n}{C!C^{n-c}} P_0, \quad C \le n \le N$$

$$P_0 = \left\{ \sum_{n=0}^{C-1} \frac{\varphi^n}{n!} + \frac{\varphi^C \left[1 - \frac{\varphi}{C}\right]^{N-C+1}}{C!\left[1 - \left(\frac{\varphi}{C}\right)\right]} \right\}^{-1} \text{ when } \frac{\varphi}{C} \ne 1$$

$$= \left\{ \sum_{n=0}^{C-1} \frac{\varphi^n}{n!} + \frac{\varphi^C}{C!}(N - C + 1) \right\}^{-1} \text{ when } \frac{\varphi}{C} = 1$$

$$L_q = P_0 \frac{\varphi^{C+1}}{(C-1)!(C-\varphi)^2} = \left[1 - \left(\frac{\varphi}{c}\right)^{N-C} - (N - C)\left(\frac{\varphi}{C}\right)^{N-C}\left(1 - \frac{\varphi}{C}\right)\right] \qquad \text{for } \frac{\varphi}{C} \ne 1$$

$$= P_0 \frac{\varphi^C (N-C)(N-C+1)}{2C!} \qquad \text{for } \frac{\varphi}{C} = 1$$

and

$$L_s = L_q + (C - \bar{C}) = L_q + \frac{\delta_{eff}}{\mu}$$

Where $\bar{C} = \sum_{n=0}^{C} (C - n) P_n$

which is the expected number of idle servers. Therefore,

$$\delta_{eff} = \delta(1 - P_N) = \mu(C - \bar{C})$$

$$W_q = \frac{\text{Lq}}{\delta_{eff}}$$

$$W_s = \frac{\text{Ls}}{\delta_{eff}}$$

Example:

In a harbour, ships arrive with a mean rate of 18 per week. The harbour has 4 docks to handle unloading and loading of ships. The service rate of individual dock is 6 per week. The arrival rate and the service rate follow Poisson distribution. At a point in time, the maximum number of ships permitted in the harbour is 6. Find

P_0 ,L_s, L_q, W_s and W_q

Solution:

We have

Arrival rate, δ= 18 per week

Service rate, μ = 6 per week

$\varphi = \frac{\delta}{\mu} = \frac{18}{6} = 3$

Now, applying the required formulae, we calculate the given quantities as follows:

$$P_0 = \left\{\sum_{n=0}^{C-1} \frac{\varphi^n}{n!} + \frac{\varphi^C\left[1-\frac{\varphi}{C}\right]^{N-C+1}}{C!\left[1-\left(\frac{\varphi}{C}\right)\right]}\right\}^{-1} \text{ when } \frac{\varphi}{C} \neq 1$$

$$= \left\{\sum_{n=0}^{4-1} \frac{3^n}{n!} + \frac{3^4\left[1-\frac{3}{4}\right]^{6-4+1}}{4!\left[1-\left(\frac{3}{4}\right)\right]}\right\}^{-1} = 0.0757$$

$$L_q = P_0 \frac{\varphi^{C+1}}{(C-1)!(C-\varphi)^2} = \left[1-\left(\frac{\varphi}{c}\right)^{N-C} - (N-C)\left(\frac{\varphi}{C}\right)^{N-C}\left(1-\frac{\varphi}{C}\right)\right] \quad \text{for } \frac{\varphi}{C} \neq 1$$

$$= 0.0757\frac{3^5}{3!(1)^2} = \left[1-\left(\frac{3}{4}\right)^{6-4} - (6-4)\left(\frac{3}{4}\right)^{6-4}\left(1-\frac{3}{4}\right)\right]$$

$= 0.479$ ship

$\bar{C} = \sum_{n=0}^{4}(4-n)P_n = 2$

Therefore,

$$P_N = \frac{\varphi^N}{C!C^{N-c}}P_0, \quad n = N$$

$P_N = \frac{3^6}{4!4^{6-4}} \times 0.0757 = 0.1437$

$\delta_{eff} = \delta(1 - P_N) = 18(1 - 0.1437) = 15.41$ ships per week

$L_s = L_q + \frac{\delta_{eff}}{\mu} = 0.479 + \frac{15.41}{6} = 3.05$ ships

$W_q = \frac{Lq}{\delta_{eff}} = \frac{0.479}{15.41} = 0.031 week = 0.217$ day

$W_s = \frac{Ls}{\delta_{eff}} = \frac{3.05}{15.41} = 0.198$ week = 1.386 day

(M/M/C): (GD/N/N) Model (for C < N)

The parameters of this model are defined below:

(i) Arrival rate follows Poisson distribution.

(ii) Service rate follows Poisson distribution.

(iii) Number of servers is C.

(iv) Service discipline is general discipline.

(v) Maximum number of customers permitted in the system is N.

(vi) Size of the calling source is N.

In this model, the following assumptions are made.

$\delta_n = (N - n)\delta, \quad 0 \leq n \leq N$

$= 0, \quad n \geq N$

$\mu_n = n\mu, \quad 0 \leq n \leq C$

$= C\mu \quad C \leq n \leq C$

$= 0 \quad n \geq N$

The steady-state formula to obtain the probability of having a customers in the system P_n and the formulas for P_0 are:

$P_n = N_n^C \varphi^n P_0 \qquad 0 \leq n \leq C$

$= N_n^C \frac{n!\varphi^n}{C!C^{n-C}} P_0 \qquad C \leq n \leq N$

$P_0 = \left\{\sum_{n=0}^{C} N_n^C \varphi^n + \sum_{n=C+1}^{N} N_n^C \frac{n!\varphi^n}{C!C^{n-C}}\right\}^{-1}$

We also have formulas for L_q and L_s as:

$L_q = \sum_{n=c+1}^{N}(n - C)P_n$

and $L_s = L_q + (C - \bar{C}) = L_q + \frac{\delta_{eff}}{\mu}$

Where $\bar{C} = \sum_{n=0}^{C}(C - n)P_n$

and $\delta_{eff} = \delta(N - L_s) = \mu(C - \bar{C})$

The formulas for $W_q = \frac{Lq}{\delta_{eff}}$ $\qquad W_s = \frac{Ls}{\delta_{eff}}$

Example:

In the machine shop of a small-scale industry, machines breakdown with a mean rate of 2 per hour. The maintenance shop of the industry has 2 mechanics who can attend the breakdown machines individually. The service rate of each of the mechanics is 1.5 machines per hour. Initially, there are 5 working machines in the machine shop. Find P_0, L_s, L_q, W_s and W_q.

Solution:

Breakdown rate of machines, δ = 2 per hour.

Service rate of individual mechanic, μ = 1.5 machines per hour.

Total number of machines in the shop, N = 5

$\varphi = \frac{\delta}{\mu} = \frac{2}{1.5} = 1.33$

So, we get

$P_0 = \left\{\sum_{n=0}^{C} N_n^C \varphi^n + \sum_{n=C+1}^{N} N_n^C \frac{n!\varphi^n}{C!C^{n-C}}\right\}^{-1}$

$= \left\{\sum_{n=0}^{2} 5_{n}^{C}(1.33)^{n} + \sum_{n=2+1}^{5} N_{n}^{C} \frac{n!(1.33)^{n}}{2!2^{n-2}}\right\}^{-1}$

= 0.0072

$L_q = \sum_{n=c+1}^{N}(n - C)P_n$

$= \sum_{n=c+1}^{N}(n - C)\, N_{n}^{C} \frac{n!\varphi^{n}}{C!C^{n-C}} P_0$

$= \sum_{n=2+1}^{5}(n - 2)\, 5_{n}^{C} \frac{n!(1.33)^{n}}{2!2^{n-2}} \times 0.0072$

= 1.604 machines

Where $\bar{C} = \sum_{n=0}^{C}(C - n)P_n = \sum_{n=0}^{2}(2 - n)P_n = 2P_0 + P_1$

= 0.06228

The other results are:

$\delta_{eff} = \mu(C - \bar{C}) = 1.5(2 - 0.06228) = 2.907$ per hour

$L_s = L_q + (C - \bar{C}) = 1.604 + (2 - 0.06228) = 3.54172$ machines

$W_q = \frac{Lq}{\delta_{eff}} = \frac{1.604}{2.907} = 0.552$ hour

$W_s = \frac{Ls}{\delta_{eff}} = \frac{3.54172}{2.907} = 1.2183$ hour

EXAMPLES:

1.The daily demand for an electronic machine is approximately 25 times every time an order is placed, a fixed cost of Rs.25 is increased. The daily holding cost per item inventory is Rs.0.40. If the lead time is 16days determine the economic lot size and the re-order point.

2. An auto industry purchases spark plugs at the rate of Rs.25 per piece. The annual consumption of spark plugs is 18,000 numbers. If the ordering cost is Rs.250 per order and carrying cost is 25 per cent per annum. What would be the EOQ? If the supplier of spark plugs offer a discount of 5 Per cent for order quantity of 3,000 numbers per order, do you accept the discount offer?

3. A company manufactures a product from a raw material, which is purchased at Rs.60 per kg. The Company incurs a handling cost of Rs.360 plus freight of Rs.390

per order. The incremental carrying cost of inventory of raw material is Rs.0.50 per kg per month. In addition, the cost of working capital finance on the investment in inventory of raw material is Rs.9 per kg per annum. The annual production of the product isn1,00,000 units and 2.5 units are obtained from one kg of raw materials.

4. Find the cost per period of individual replacement policy of an installation of 300 electric bulbs given the following

(i) Cot of replacing individual bulb is Rs.3

(ii) Conditional probability of failure is given below:

Week	0	1	2	3	4
Conditional Probability of failure	0	1/10	1/3	2/3	1

5. A factory has large number of bulbs, all of which must be in working condition. The mortality of bulbs is given in the following table:

week	1	2	3	4	5	6
Proportion of Bulbs failing	0.10	0.15	0.25	0.35	0.12	0.03

If a bulb fails in service, it costs Rs. 3.50 to be replaced; but if all the bulbs are replaced at a time it costs Rs. 1.20 each, find the optimum group replacement policy.

6. Write short notes on applications of Queuing Model.

7. Write Classification of Queuing Model.

8. Write characteristics of Queuing System.

9. There are four booking counters in a railway station, the arrival rate of customers follows Poisson distribution and it is 30 per hour. The service rate also follows poisson distribution and it is 10 customers per hour. Find the following:

(i) Average number of customers waiting in the system.

(ii)Average waiting time of a customer in the system.

10. Ships arrive at a port at the rate of one in every 4 hours with exponential distribution of inter arrival times. The time a ship occupies a berth for unloading has exponential distribution with an average of 10 hours. If the average delay of ships waiting for berths is to kept below 14 hours, how many berths should be provided at the port?

UNIT V

GAME THEORY AND STRATEGIES

INTRODUCTION

Games theory – two player zero sum game theory – Saddle Point –Mixed Strategies for games without saddle points – Dominance method – Algebraic & Graphical Methods. Decision Making under risk and uncertainty; Maximax, Maximin, Regret Hurwitz and Laplace Criteria in Business and Decision Making

5.1 Introduction to game theory

Game theory seeks to analyze competing situations which arise out of conflicts of interest. Abraham Maslow's hierarchical model of human needs lays emphasis on fulfilling the basic needs such as food, water, clothes, shelter, air, safety and security. There is conflict of interest between animals and plants in the consumption of natural resources. Animals compete among themselves for securing food. Man competes with animals to earn his food. A man also competes with another man. In the past, nations waged wars to expand the territory of their rule. In the present day world, business organizations compete with each other in getting the market share. The conflicts of interests of human beings are not confined to the basic needs alone. Again considering Abraham Maslow's model of human needs, one can realize that conflicts also arise due to the higher levels of human needs such as love, affection, affiliation, recognition, status, dominance, power, esteem, ego, self-respect, etc. Sometimes one witnesses clashes of ideas of intellectuals also. Every intelligent and rational participant in a conflict wants to be a winner but not all participants can be the winners at a time. The situations of conflict gave birth to Darwin's theory of the 'survival of the fittest'.

5.2 Assumptions for a Competitive Game

Game theory helps in finding out the best course of action for a firm in view of the anticipated countermoves from the competing organizations. A competitive situation is a competitive game if the following properties hold:

1. The number of competitors is finite, say N.
2. A finite set of possible courses of action is available to each of the N competitors.
3. A play of the game results when each competitor selects a course of action from the setof courses available to him. In game theory we make an important assumption that al the players select their courses of action simultaneously. As a result, no competitor willbe in a position to know the choices of his competitors.
4. The outcome of a play consists of the particular courses of action chosen by the individual players. Each outcome leads to a set of payments, one to each player, whichmay be either positive, or negative, or zero.

5.3 Managerial Applications of the Theory of Games

The techniques of game theory can be effectively applied to various managerial problemsas detailed below:

1) Analysis of the market strategies of a business organization in the long run.
2) Evaluation of the responses of the consumers to a new product.
3) Resolving the conflict between two groups in a business organization.
4) Decision making on the techniques to increase market share.
5) Material procurement process.
6) Decision making for transportation problem.
7) Evaluation of the distribution system.
8) Evaluation of the location of the facilities.
9) Examination of new business ventures and
10) Competitive economic environment.

5.4 Definitions

The competitors or decision makers in a game are called the players of the game.

Strategies:

The alternative courses of action available to a player are referred to as his strategies.

Pay off:

The outcome of playing a game is called the pay off to the concerned player.

Optimal Strategy:

A strategy by which a player can achieve the best pay off is called the optimal strategy for him.

Zero-sum game:

A game in which the total payoffs to all the players at the end of the game is zero is referred to as a zero-sum game.

Non-zero sum game:

Games with "less than complete conflict of interest" are called non-zero sum games. The problems faced by a large number of business organizations come under this category. In such games, the gain of one player in terms of his success need not be completely at the expense of the other player.

Payoff matrix:

The tabular display of the payoffs to players under various alternatives is called the payoff matrix of the game.

Pure strategy:

If the game is such that each player can identify one and only one strategy as the optimal strategy in each play of the game, then that strategy is referred to as the best strategy for that player and the game is referred to as a game of pure strategy or a pure game.

Mixed strategy:

If there is no one specific strategy as the 'best strategy' for any player in a game, then the game is referred to as a game of mixed strategy or a mixed game. In such a game, each player has to choose different alternative courses of action from time to time.

N-person game:

A game in which N-players take part is called an N-person game.

Maximin-Minimax Principle:

The maximum of the minimum gains is called the maximin value of the game and the corresponding strategy is called the maximin strategy. Similarly the minimum of the maximum losses is called the minimax value of the game and the corresponding strategy is called the minimax strategy. If both the values are equal, then that would guarantee the best of the worst results.

Saddle point:

A saddle point of a game is that place in the payoff matrix where the maximum of the row minima is equal to the minimum of the column maxima. The payoff at the saddle point is called **the value of the game** and the corresponding strategies are called the **pure strategies. Dominance**:

One of the strategies of either player may be inferior to at least one of the remaining ones. The superior strategies are said to dominate the inferior ones.

Types of Games:

There are several classifications of a game. The classification may be based on various factors such as the number of participants, the gain or loss to each participant, the number of strategies available to each participant, etc. Some of the important types of games are enumerated below.

Two person games and n-person games:

In two person games, there are exactly two players and each competitor will have a finite number of strategies. If the number of players in a game exceeds two, then we refer to the game as n-person game.

Zero sum game and non-zero sum game:

If the sum of the payments to all the players in a game is zero for every possible outcome of the game, then we refer to the game as a zero sum game. If the sum of the payoffs from any play of the game is eitherpositive or negative but not zero, then the game is called a non-zero sum game

2x2 two person game and 2xn and mx2 games:

When the number of players in a game is two and each player has exactly two strategies, the game is referredto as 2x2 two person game.

A game in which the first player has precisely two strategies and the second player has three or morestrategies is called an 2xn game.

A game in which the first player has three or more strategies and the second player has exactly two strategies is called an mx2 game.

3x3 and large games:

When the number of players in a game is two and each player has exactly three strategies, we call it a 3x3 two person game.Two-person zero sum games are said to be larger if each of the two players has 3 or morechoices.The examination of 3x3 and larger games is involves difficulties. For such games, the technique oflinear programming can be used as a method of solution to identify the optimum strategies for the two players.

Two-person zero sum game

A game with only two players, say player A and player B, is called a two-person zero sum game if thegain of the player A is equal to the loss of the player B, so that the total sum is zero.

Minimax and Maximin Principles

The selection of an optimal strategy by each player without the knowledge of the competitor's strategy is the basic problem of playing games.

The objective of game theory is to know how these players must select their respective strategies, so that they may optimize their payoffs. Such a criterion of decision making is referred to as minimax-maximin principle. This principle in games of pure strategies leads to the best possible selection of a strategy for both players.

For example, if player A chooses his i[th] strategy, then he gains at least the payoff min a

, which is minimum of the i[th] row elements in the payoff matrix. Since his objective is to maximize his payoff, he can choose strategy *i* so as to make his payoff as large as possible.

i.e., a payoff which is not less than max min a_{ij} . 1

Similarly player B can choose j[th] column elements so as to make his loss not greater than min max a .

If the maximin value for a player is equal to the minimax value for another player then the game is said to have a saddle point (equilibrium point) and the corresponding strategies are called optimal strategies. If there are two or more saddle points, they must be equal.

The amount of payoff, i.e., *V* at an equilibrium point is known as the **value of the**

The optimal strategies can be identified by the players in the long run.

Fair game:

The game is said to be fair if the value of the game *V = 0.*

Example 1:

Solve the game with the following pay-off matrix.

Player B

Strategies

		I	II	III	IV	V
Player A Strategie	1	-2	5	- 3	6	7
	2	4	6	8	-1	6
	3	8	2	3	5	4
	4	15	14	18	12	20

Solution:

First consider the minimum of each row.

Row	Minimum Value
1	-3
2	-1
3	2
4	12

Maximum of {-3, -1, 2, 12} = 12

Next consider the maximum of each column.

Column	Maximum Value
1	15
2	14
3	18
4	12
5	20

Minimum of {15, 14, 18, 12, 20}= 12

We see that the maximum of row minima = the minimum of the column maxima. So the game has a saddle point. The common value is 12. Therefore the value V of the game = 12.

Interpretation:

In the long run, the following best strategies will be identified by the two players:

The best strategy for player A is strategy 4.

The best strategy for player B is strategy IV.

The game is favourable to player A.

Example 2:

Solve the game with the following pay-off matrix

Player Y

Strategies

		I	II	III	IV	V
Player X Strategies	1	9	12	7	14	26
	2	25	35	20	28	30
	3	7	6	–8	3	2
	4	8	11	13	–2	1

Solution:

First consider the minimum of each row.

Row	Minimum Value
1	7
2	20
3	-8
4	-2

Maximum of {7, 20, –8, -2} = 20

Next consider the maximum of each column.

Column	Maximum Value
1	25
2	35
3	20

4	28
5	30

Minimum of {25, 35, 20, 28, 30}= 20

It is observed that the maximum of row minima and the minimum of the column maxima are equal. Hence the given the game has a saddle point. The common value is 20. This indicates that the value V of the game is 20.

Interpretation.

The best strategy for player X is strategy 2. The best strategy for player Y is strategy III.

The game is favourable to player A.

Example 3:

Solve the following game:

Player B Strategies

Player A Strategies		*I*	*II*	*III*	*IV*
	1	1	–6	8	4
	2	3	–7	2	–8
	3	5	–5	–1	0
	4	3	–4	5	7

Solution

First consider the minimum of each row.

Row	Minimum Value
1	-6
2	-8
3	-5
4	-4

Maximum of {-6, -8, -5, -4} = -4

Next consider the maximum of each column.

Column	Maximum Value
1	5
2	-4
3	8
4	7

Minimum of {5, -4, 8, 7}= - 4

Since the max {row minima} = min {column maxima}, the game under consideration has asaddle point. The common value is –4. Hence the value of the game is –4.

Interpretation. The best strategy for player A is strategy 4.The best strategy for player B is strategy II. Since the value of the game is negative, it is concluded that the game is favorable to player B.

5.5 Games with Mixed Strategies

In certain cases, no pure strategy solutions exist for the game. In other words, saddle point does not exist. In all such game, both players may adopt an optimal blend of the strategies called Mixed Strategy to find a saddle point. The optimal mix for each player may be determined by assigningeach strategy a probability of it being chosen. Thus these mixed strategies are probabilistic combinations of available better strategies and these games hence called Probabilistic games.

The probabilistic mixed strategy games without saddle points are commonly solved by any of thefollowing methods

Sl. No.	Method	Applicable to
1	Analytical Method	2x2 games
2	Graphical Method	2x2, mx2 and 2xn games

A 2 x 2 payoff matrix where there is no saddle point can be solved by analytical method.Given the matrix

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Value of the game is

$$V = \frac{a_{11}\,a_{22} - a_{21}\,a_{12}}{(a_{11}+a_{22}) - (a_{12}+a_{21})}$$

$$x_1 = \frac{a_{22} - a_{21}}{(a_{11}+a_{22}) - (a_{12}+a_{21})}, \quad x_2 = \frac{a_{11} - a_{12}}{(a_{11}+a_{22}) - (a_{12}+a_{21})}$$

$$y_1 = \frac{a_{22} - a_{12}}{(a_{11}+a_{22}) - (a_{12}+a_{21})}, \quad y_2 = \frac{a_{11} - a_{21}}{(a_{11}+a_{22}) - (a_{12}+a_{21})}$$

Example : Solve

Solution

$$A\ \overset{B}{\begin{bmatrix} 5 & 1 \\ 3 & 4 \end{bmatrix}} \qquad A\ \overset{B}{\begin{bmatrix} 5 & 1 \\ 3 & 4 \end{bmatrix}}$$

It is a 2 x 2 matrix and no saddle point exists. We can solve by analytical method

$$V = \frac{a_{11}a_{22} - a_{21}a_{12}}{(a_{11}+a_{22}) - (a_{12}+a_{21})} = \frac{20-3}{9-4}$$

V = 17 / 5

$$x_1 = \frac{a_{22} - a_{21}}{(a_{11}+a_{22}) - (a_{12}+a_{21})}, \quad x_2 = \frac{a_{11} - a_{12}}{(a_{11}+a_{22}) - (a_{12}+a_{21})}$$

SA = (P1, P2) = (1/5, 4 /5)SB = (q1, q2) = (3/5, 2 /5)

$$y_1 = \frac{a_{22} - a_{12}}{(a_{11}+a_{22}) - (a_{12}+a_{21})}, \quad y_2 = \frac{a_{11} - a_{21}}{(a_{11}+a_{22}) - (a_{12}+a_{21})}$$

Example: Solve the given matrix

$$A\ \overset{B}{\begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix}}$$

Solution

$$\begin{array}{c c} & B \\ A & \begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix} \end{array}$$

$$V = \frac{a_{11}\, a_{22} - a_{21}\, a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} = \frac{0 - 1}{2 + 2}$$

V = - 1 / 4
SA = (P_1, P_2) = (1/4, 3 /4)SB = (q_1, q_2) = (1/4, 3 /4)

5.6 Graphical method

The graphical method is used to solve the games whose payoff matrix has

- Two rows and n columns (2 x n)
- m rows and two columns (m x 2)

Algorithm for solving 2 x n matrix games

- Draw two vertical axes 1 unit apart. The two lines are x_1 = 0, x_1 = 1
- Take the points of the first row in the payoff matrix on the vertical line x_1 = 1 and the pointsof the second row in the payoff matrix on the vertical line x_1 = 0.
- The point a_{1j} on axis x_1 = 1 is then joined to the point a_{2j} on the axis x_1 = 0 to give a straightline. Draw 'n' straight lines for j=1, 2… n and determine the highest point of the lower envelope obtained. This will be the **maximini point**.
- The two or more lines passing through the maximini point

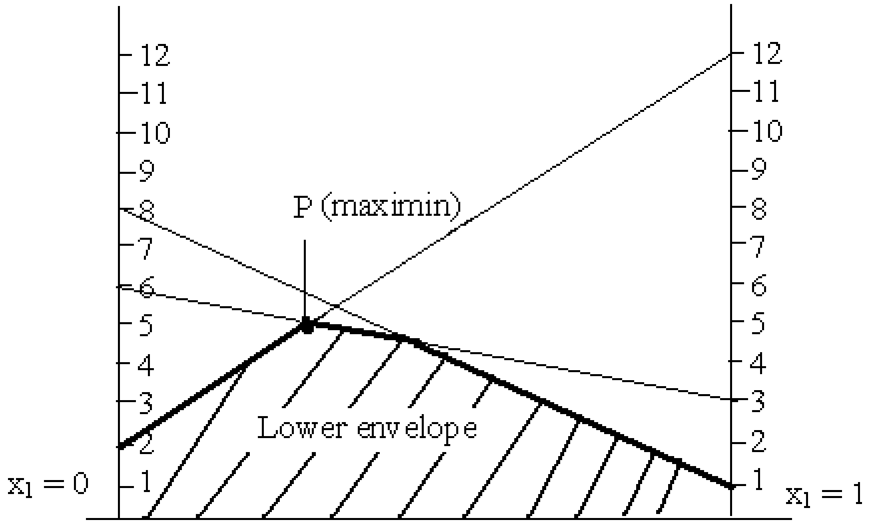
determines the required 2 x 2 payoff matrix. This in turn gives the optimum solution by making use of analytical method.

Example 1

Solve by graphical method

$$\begin{array}{c} \\ A1 \\ A2 \end{array} \begin{array}{c} \begin{array}{ccc} B1 & B2 & B3 \end{array} \\ \begin{bmatrix} 1 & 3 & 12 \\ 8 & 6 & 2 \end{bmatrix} \end{array}$$

Solution



$$\begin{array}{c} \\ A1 \\ A2 \end{array} \begin{array}{c} \begin{array}{cc} B2 & B3 \end{array} \\ \begin{bmatrix} 3 & 12 \\ 6 & 2 \end{bmatrix} \end{array}$$

$$V = \frac{a_{11}\, a_{22} - a_{21}\, a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} = \frac{6 - 72}{5 - 18}$$

V = 66/13

SA = (4/13, 9 /13)

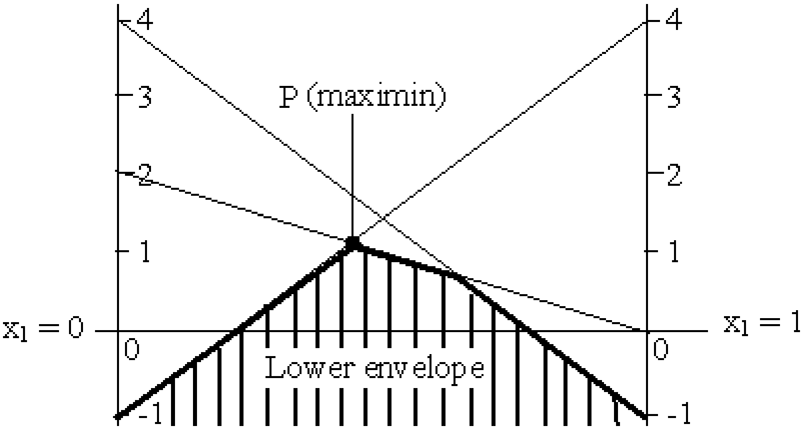
SB = (0, 10/13, 3 /13)

Example 2

	B1	B2	B3
A1	4	-1	0
A2	-1	4	2

Solve by graphical method

Solution



	B1	B3
A1	4	0
A2	-1	2

$$V = \frac{a_{11}\,a_{22} - a_{21}\,a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} = \frac{8 - 0}{6 + 1}$$

V = 8/7

Algorithm for solving m x 2 matrix games

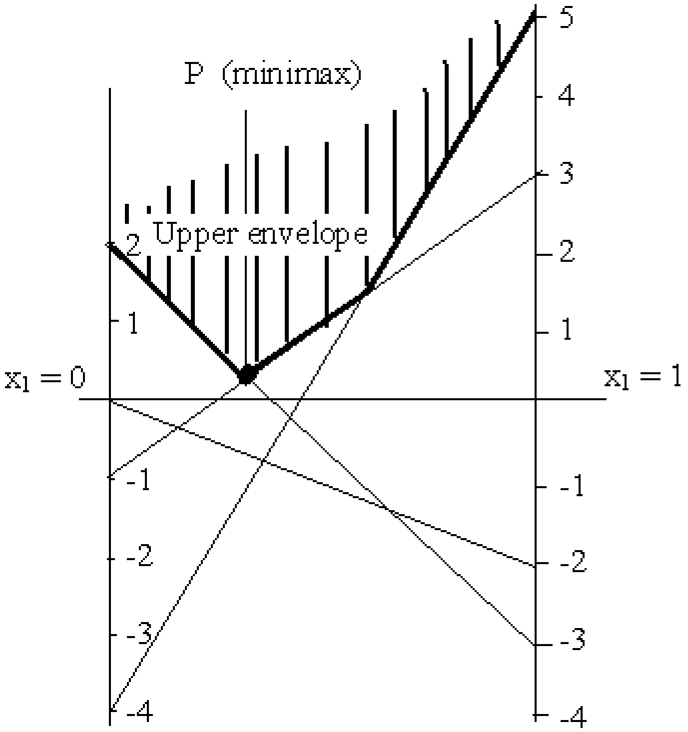
- Draw two vertical axes 1 unit apart. The two lines are $x_1 = 0$, $x_1 = 1$
- Take the points of the first row in the payoff matrix on the vertical line $x_1 = 1$ and the pointsof the second row in the payoff matrix on the vertical line $x_1 = 0$.
- The point a_{1j} on axis $x_1 = 1$ is then joined to the point a_{2j} on the axis $x_1 = 0$ to give a straightline. Draw 'n' straight lines for j=1, 2… n and determine the lowest point of the upper envelope obtained. This will be the **minimax point**.
- The two or more lines passing through the minimax point determines the required 2 x 2 payoff matrix. This in turn gives the optimum solution by making use of analytical method.

Example 1

Solve by graphical method

	B1	B2
A1	-2	0
A2	3	-1
A3	-3	2
A4	5	-4

Solution



	B1	B2
A2	3	-1
A3	-3	2

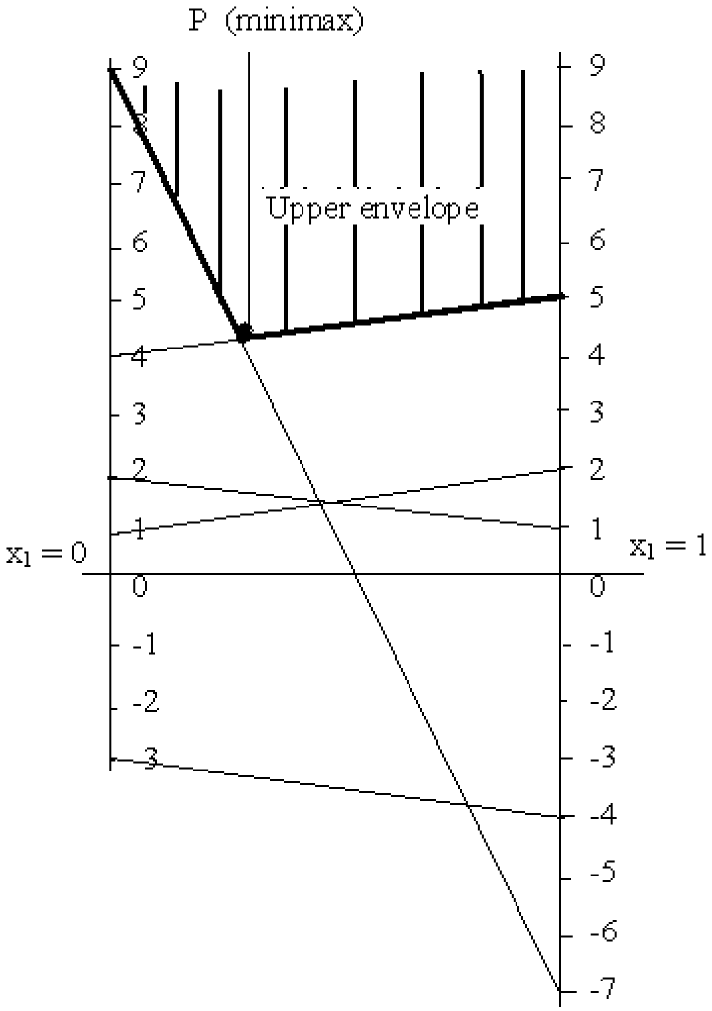
$$V = \frac{a_{11}\,a_{22} - a_{21}\,a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} = \frac{6 - 3}{5 + 4}$$

V = 3/9 = 1/3

Example 2

Solve by graphical method

	B1	B2
A1	1	2
A2	5	4
A3	-7	9
A4	-4	-3
A5	2	1



Solution

$$\begin{array}{c} \\ A2 \\ A3 \end{array} \begin{array}{c} \begin{array}{cc} B1 & B2 \end{array} \\ \begin{bmatrix} 5 & 4 \\ -7 & 9 \end{bmatrix} \end{array}$$

$$V = \frac{a_{11}\, a_{22} - a_{21}\, a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} = \frac{45 + 28}{14 + 3}$$

V = 73/17
SA = (0, 16/17, 1/17, 0, 0)
SB = (5/17, 12 /17)

5.7 DOMINANCE PROPERTY

Principle of dominance is applicable to both pure strategies and mixed strategies. Sometimes, it is observed that one of the pure strategies of either players is always inferior to at least one of the remaining strategies. The superior strategies are said to dominate the inferior ones. The player would have no incentive to use inferior strategies which are dominated by the superior ones. In such cases of dominance, the size of the pay-off matrix by deleting those strategies which are dominated by the others.

The dominance properties are*:*

1. If all the elements of a row say K^{th}, are less than or equal to the corresponding elements of any other row (say r^{th} row), then k^{th} row is dominated by the r^{th} row.
2. If all the elements of column, say k^{th}, are greater than or equal to the corresponding elements of any other column, say r^{th} then K^{th} column is dominated by the r^{th} row.
3. A pure strategy may be dominated, if it is inferior to average of two or more other pure strategies.

Example : *Solve the following game*

Player *B*

1	2	3	4

Player A				
1	3	2	4	0
2	3	4	2	4
3	4	2	4	0
4	0	4	0	8

Solution: Since all the elements of 3rd row are greater than or equal to the correspond-ing elements of 1st row, third (3rd) row is dominating 1st row and hence row 1 can be elimi-nated.

	Player B 1	2	3	4
Player A 1	3	4	2	4
3	4	2	4	0
4	0	4	0	8

Again, all the elements of the 1st column are greater than or equal to the corresponding elements of the 3rd column.

□ 3rd column is dominating the first column. Eliminating 1st column, the matrix is reduced to

	Player B 2	3	4
Player A 2	4	2	4
3	2	4	0
4	4	0	8

Here, the linear combination of 2nd and 4th column dominates the 2nd

Eliminating 1st column, the reduced matrix becomes,

		Player *B* 3	4
Player *A*	2	2	4
	3	4	0
	4	0	8

Here again, the convex linear combination of 3 and 4 of player '*A*' dominate 2nd because

Eliminating 1st row, the reduced matrix becomes

		Player *B* 3	4
Player *A*	3	4	0
	4	0	8

the probabilities of mixed strategies for player "A" and'*B*'. Thus, optimum strategies are:

For player *A*, [0, 0, 2/3, 1/3]; For player *B*, [0, 0, 2/3, 1/3] and The value of the game is

$$\frac{a_{11}a_{22}}{a_{21}a_{12}} = \frac{4 \times 8 + 0 \times 0}{12}$$

= 32/12

Example: *Solve the following game by using the principle of dominance*

Player B

		B_1	B_2	B_3	B_4	B_5	B_6
	A_1	4	2	0	2	1	1
	A_2	4	3	1	3	2	2
Player A	A_3	4	3	7	–5	1	2
	A_4	4	3	4	–1	2	2
	A_5	4	3	3	–2	2	2

Solution: The pay-off matrix has no saddle point.

All the elements of row A_1 are dominated by row A_2 and row A_5 is dominated by row A_4. Hence row A_1 and A_5 can be eliminated. Hence, the pay-off matrix is reduced to

Player B

		B_1	B_2	B_3	B_4	B_5	B_6
	A_2	4	3	1	3	2	2
Player A	A_3	4	3	7	–5	1	2
	A_4	4	3	4	–1	2	2

From player B's point of view, column B_1 and B_2 are dominated by columns B_4, B_4 and B_6 and column B_6 is dominated by column B_5. Hence strategies B_1, B_2 and B_6 are elimi- nated. The modified pay-off matrix is

Player B

		B_3	B_4	B_5
	A_2	1	3	3
Player	A_3	7	–5	1
	A_4	4	–1	2

Now, none of the single row or column dominates another row or column *i.e.* none of the pure strategies of A and B is inferior to any of the other strategies.

This strategy of B is superior to strategy B_5 because the B_5 strategy will result him in greater loss. So, the strategy B_5 can be eliminated. The modified matrix is

Player B

B_3 B_4

	A_2	1	3
Player A	A_3	7	–5
	A_4	4	–1

Thus the resulting matrix (2 × 2) is given by

Player B

B_3 B_4

A:A2,A3

Solving this (2 × 2) game is

Optimal strategy for A – [0, 6/7, 1/7, 0, 0] Optimal strategy for B [0, 0, 4/7, 3/7, 0, 0]

The value of the game to player A is $\frac{13}{7}$.

Let us sum up

The concept of 'strategy' refers to a comprehensive collection of action plans that clearly outline how a player intends to respond to every conceivable scenario that may arise during a game. In other words, a player's strategy serves as the guideline they follow when selecting from their available options. Strategies can be divided into two categories: pure strategy and mixed strategy.

Payoff refers to the results obtained from participating in a game. A payoff matrix is a structured table that illustrates the rewards received by the player listed on the left side, based on the various possible outcomes throughout the course of the game. Payments originate from the player specified at the top of the table.

In two-person games, although each player may have multiple choices for their moves, only two players are involved, which is why it is classified as a two-person game. When there are more than two participants, the game is usually referred to as an n-person game. A zero-sum game is characterized by the fact that the total payments to all players equal zero for every possible outcome. In these scenarios, the points gained by one player correspond precisely to the points lost by the other. A two-person zero-sum game specifically refers to a situation where one player's gain is matched by the other player's loss. This type of game is often described as a rectangular game due to the rectangular shape of the payoff matrix.

A saddle point within the payoff matrix is where the highest value among the row's minimums aligns with the lowest value among the column's maximums. The payoff corresponding to this saddle point is known as the 'value of the game'. It is typically observed that one of the pure strategies for either player will always be less effective than at least one of the other available strategies.

The strategies that perform better are said to dominate those that are inferior. In these instances, we can simplify the payoff matrix by eliminating the dominated strategies. The maximin-minimax principle serves as a guide for selecting the optimal strategies for both players.

Check Your Progress

1. How is the optimal strategy mixture determined by each player?

2. What are the characteristics of two-person zero-sum game?

3. What do you understand by the value of game?

SELF ASSESSMENT QUESTIONS

1. Solve the following pay-off matrix:

Player A	**Player B**			
	Strategies	I	II	III
	I	6	8	6
	II	4	12	2

2. Solve the following pay-off matrix:

Player A	**Player B**					
	Strategies	I	II	III	IV	V
	I	9	3	1	8	0
	II	6	5	4	6	7
	III	2	4	3	3	8
	IV	5	6	2	2	1

3. Solve the following pay-off matrix:

Player A	**Player B**		
	Strategy	I	II
	I	1	5
	II	4	2

GOSSARY : Pure strategy: A strategy is called pure if one knows in advance of the play that it is certain to be adopted, irrespective of the strategy the other players might choose.

Zero-sum game: A zero-sum game is one in which the sum of the payment to all the competitors is zero, for every possible outcome of the game is in a game if the sum of the points won, equals the sum of the points lost.

Two-person zero-sum game: A game with two players, where the gain of one player equals the loss of the other, is known as a two-person zero-sum game.

Saddle point: A saddle point is a position in the payoff matrix where, the maximum of row minima coincides with the minimum of column maxima.

Exercise:

1. Define saddle point.
2. Define types of game.
3. Write the Applications of game theory.
4. Write short notes on Advantages and disadvantages of game theory.
5. Find the value of the following game:

		A	B
		1	2
A	1	6	9
B	2	8	4

6. Write short notes on Monte Carlo Simulation.
7. Give various types of simulation.
8. What are the applications of simulation?
9. Consider the payoff matrix with respect to the player A as shown in table. Solve this game optimally using graphical method.

	1	2	3	4	5
1	4	2	1	7	3
2	2	7	8	1	5

10. Differentiate pure strategy and mixed strategy.